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No Self-Interaction for Two-Column Massless Fields

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Abstract

We investigate the problem of introducing consistent self-couplings in free theories for mixed tensor gauge fields whose symmetry properties are characterized by Young diagrams made of two columns of arbitrary (but different) lengths. We prove that, in flat space, these theories admit no local, Poincaré-invariant, smooth, self-interacting deformation with at most two derivatives in the Lagrangian. Relaxing the derivative and Lorentz-invariance assumptions, there still is no deformation that modifies the gauge algebra, and in most cases no deformation that alters the gauge transformations. Our approach is based on a BRST-cohomology deformation procedure.

[1] "Aspirant du F.N.R.S., Belgium"

1 Introduction

These last few years, mixed symmetry gauge fields (*i.e.* that are neither completely symmetric nor antisymmetric) have attracted some renewed attention [1–10], thereby reviving the efforts made in this direction during the eighties, under the prompt of string field theory [11–14]. Mixed-symmetry fields appear in a wide variety of higher-dimensional ($D > 4$) contexts. Indeed, group theory imposes that first-quantized particles propagating in flat background should provide representations of the Poincaré group. The cases $D = 3\,,4$ are very particular in the sense that each tensor irreducible representation (irrep.) of the little groups $O(2)$ and $O(3)$ is equivalent to a completely symmetric tensor irrep. (pictured by a one-row Young diagram with S columns for a spin-S particle). When $D > 4$, more complicated Young diagrams are allowed. For instance, all critical string theory spectra contain massive fields in mixed symmetry representations of the Lorentz group. In the tensionless limit ($\alpha' \rightarrow \infty$) all these massive excitations become massless. Another way to generate various mixed symmetry fields is by dualizing totally symmetric fields in higher dimensions [3,6].

An irrep. of the general linear group $GL(D,\mathbb{R})$ is denoted by $[c_1, c_2, \ldots, c_L]$, where c_i indicates the number of boxes in the i-th column of the Young diagram characterizing the corresponding irrep. We will focus on theories describing gauge fields $\phi_{\mu_1\ldots\mu_p|\nu_1\ldots\nu_q}$ whose symmetries correspond to the Young diagram $[p,q]$ formed by two columns of arbitrary (but different) lengths p and q ($p > q$). The physical degrees of freedom for such theories correspond to a traceless tensor carrying an irrep. of the little group $O(D-2)$ associated with the Young diagram $[p,q]$. Therefore, we will work in spacetime dimension $D \geqslant p+q+2$ so that the field carries local physical degrees of freedom. Such fields were studied recently at the free level in AdS background [9, 10]. In the sequel, we will frequently use a loose terminology by referring to a tensor irrep. by its Young diagram.

In the present paper, we address the natural problem of switching on consistent self-interactions among $[p,q]$-type tensor gauge fields in flat background, where $p \neq q$. As in [15–20], we use the BRST-cohomological reformulation of the Noether method for the problem of consistent interactions [21]. For an alternative Hamiltonian-based deformation point of view, see [22]. The question of consistent self-interactions in flat background has already been investigated in the case of vector (*i.e.* $[1,0]$) gauge fields in [15], p-forms (*i.e.* $[p,0]$-fields) in [16], Fierz-Pauli $[1,1]$-fields in [17], $[p,1]$-fields ($p > 1$) in [18], $[2,2]$-fields in [19] and $[p,p]$-fields ($p > 1$) in [20]. Here, we extend and strengthen the results of [18] by relaxing some assumptions on the number of derivatives in the interactions. The present work is thus the completion of the analysis of self-interactions for *arbitrary* $[p,q]$-type tensor gauge fields in flat space.

Our main (no-go) result can be stated as follows, spelling out explicitly our assumptions:

Theorem: *In flat space and under the assumptions of locality and translation-invariance, there is no consistent smooth deformation of the free theory for $[p,q]$-type tensor gauge fields with $p \neq q$ that modifies the gauge algebra. Furthermore, for $q > 1$, when there is no positive integer n such that $p+2 = (n+1)(q+1)$, there exists no smooth deformation that alters the gauge transformations either. Finally, if one excludes deformations that involve four derivatives or more in the Lagrangian and that are not Lorentz-invariant, then there is no smooth deformation at all.*

The paper is organized as follows. In Section 2, we review the free theory of $[p,q]$-type tensor gauge fields. In Section 3, we introduce the BRST construction for the theory. Sections 4 to 7 are devoted to the proof of cohomological results. We compute $H(\gamma)$ in Section 4, an invariant Poincaré lemma is proved in Section 5, the cohomologies $H^D_k(\delta|d)$ and $H^{D\,inv}_k(\delta|d)$ are computed respectively in Sections 6 and 7. The self-interaction question is answered in Section 8. A brief concluding section is finally followed by three appendices containing the proofs of three theorems presented in the core of the paper.

2 Free theory

As stated above, we consider theories for mixed tensor gauge fields $\phi_{\mu_1\ldots\mu_p|\nu_1\ldots\nu_q}$ whose symmetry properties are characterized by two columns of arbitrary (but different) lengths. In other words, the gauge field obeys the conditions

$$\phi_{\mu_1\ldots\mu_p|\nu_1\ldots\nu_q} = \phi_{[\mu_1\ldots\mu_p]|\nu_1\ldots\nu_q} = \phi_{\mu_1\ldots\mu_p|[\nu_1\ldots\nu_q]}\,,$$
$$\phi_{[\mu_1\ldots\mu_p|\nu_1]\nu_2\ldots\nu_q} = 0\,,$$

where square brackets denote strength-one complete antisymmetrisation.

2.1 Lagrangian and gauge invariances

The Lagrangian of the free theory is

$$\mathcal{L} = -\frac{1}{2\,(p+1)!\,q!}\,\delta^{[\rho_1\ldots\rho_q\mu_1\ldots\mu_{p+1}]}_{[\nu_1\ldots\nu_q\sigma_1\ldots\sigma_{p+1}]}\,\partial^{[\sigma_1}\phi^{\sigma_2\ldots\sigma_{p+1}]|}{}_{\rho_1\ldots\rho_q}\,\partial_{[\mu_1}\phi_{\mu_2\ldots\mu_{p+1}]|}{}^{\nu_1\ldots\nu_q}\,,$$

where the generalized Kronecker delta has strength one. This Lagrangian was obtained for $[2,1]$-fields in [11], for $[p,1]$-fields in [12] and, for the general case of $[p,q]$-fields, in the second paper of [5].

The quadratic action

$$S_0[\phi] = \int d^Dx\,\mathcal{L}(\partial\phi) \tag{2.1}$$

is invariant under gauge transformations with gauge parameters $\alpha^{(1,0)}$ and $\alpha^{(0,1)}$ that have respective symmetries $[p-1,q]$ and $[p,q-1]$. In the same manner as p-forms, these gauge transformations are *reducible*, their order of reducibility growing with p. We identify the

gauge field ϕ with $\alpha^{(0,0)}$, the zeroth order parameter of reducibility. The gauge transformations and their reducibilities are[2]

$$\begin{aligned}\delta\alpha^{(i,j)}_{\mu_{[p-i]}|\nu_{[q-j]}} &= \partial_{[\mu_1}\alpha^{(i+1,j)}_{\mu_2\ldots\mu_{p-i}]|\,\nu_{[q-j]}} \\ &+ b_{i,j}\left(\alpha^{(i,j+1)}_{\mu_{[p-i]}|\,[\nu_{[q-j-1]},\nu_{q-j}]} + a_{i,j}\,\alpha^{(i,j+1)}_{\nu_{[q-j]}[\mu_{q-j+1}\ldots\mu_{p-i}|\,\mu_{[q-j-1]},\mu_{q-j}]}\right) \end{aligned} \quad (2.2)$$

where $i = 0, ..., p-q$ and $j = 0, ..., q$. The coefficients $a_{i,j}$ and $b_{i,j}$ are given by

$$a_{i,j} = \frac{(p-i)!}{(p-i-q+j+1)!\,(q-j)!}\,, \quad b_{i,j} = (-)^i\,\frac{(p-q+j+2)}{(p-i-q+j+2)}\,.$$

To the above formulae, we must add the convention that, for all j, $\alpha^{(p-q+1,j)} = 0 = \alpha^{(j,q+1)}$. The symmetry properties of the parameters $\alpha^{(i,j)}$ are those of Young diagrams with two columns of lengths $p-i$ and $q-j$. More details on the reducibility parameters $\alpha^{(i,j)}_{\mu_1\ldots\mu_{p-i}|\,\nu_1\ldots\nu_{q-j}}$ will be given in Subsection 3.2.

The fundamental gauge-invariant object is the field strength K, the $[p+1, q+1]$-tensor defined as the double curl of the gauge field

$$K_{\mu_1\ldots\mu_{p+1}|\,\nu_1\ldots\nu_{q+1}} \equiv \partial_{[\mu_1}\phi_{\mu_2\ldots\mu_{p+1}]\,|\,[\nu_1\ldots\nu_q\,,\,\nu_{q+1}]}\,.$$

By definition, it satisfies the Bianchi (BII) identities

$$\partial_{[\mu_1}K_{\mu_2\ldots\mu_{p+2}]|\,\nu_1\ldots\nu_{q+1}} = 0\,, \quad K_{\mu_1\ldots\mu_{p+1}\,|\,[\nu_1\ldots\nu_{q+1},\nu_{q+2}]} = 0\,. \quad (2.3)$$

The field strength tensor K plays a crucial role in the determination of the physical degrees of freedom described by the action $S_0[\phi]$.

2.2 Equations of motion

The equations of motion are expressed in terms of the field strength:

$$G^{\mu_1\ldots\mu_p|}{}_{\nu_1\ldots\nu_q} \equiv \frac{\delta\mathcal{L}}{\delta\phi_{\mu_1\ldots\mu_p|}{}^{\nu_1\ldots\nu_q}} = \frac{1}{(p+1)!q!}\,\delta^{[\rho_1\ldots\rho_{q+1}\mu_1\ldots\mu_p]}_{[\nu_1\ldots\nu_q\sigma_1\ldots\sigma_{p+1}]}\,K^{\sigma_1\ldots\sigma_{p+1}|}{}_{\rho_1\ldots\rho_{q+1}} \approx 0\,,$$

where a weak equality "$\approx$" means "equal on the surface of the solutions of the equations of motion". This is a generalization of vacuum Einstein equations, linearized around the flat background. Taking successive traces of the equations of motion, one can show that they are equivalent to the tracelessness of the field strength

$$\eta^{\sigma_1\rho_1}K_{\sigma_1\ldots\sigma_{p+1}|\,\rho_1\ldots\rho_{q+1}} \approx 0\,. \quad (2.4)$$

This equation generalizes the vanishing of the Ricci tensor (in the vacuum), and is non-trivial only when $p+q+2 \leqslant D$. Together with the "Ricci equation" (2.4), the Bianchi identities (2.3) imply [3]

$$\partial^{\sigma_1}K_{\sigma_1\ldots\sigma_{p+1}|\,\rho_1\ldots\rho_{q+1}} \approx 0 \approx \partial^{\rho_1}K_{\sigma_1\ldots\sigma_{p+1}|\,\rho_1\ldots\rho_{q+1}}\,. \quad (2.5)$$

[2]We introduce the short notation $\mu_{[p]} \equiv [\mu_1\ldots\mu_p]$. A comma stands for a derivative: $\alpha_{,\nu} \equiv \partial_\nu\alpha$.

The gauge invariance of the action is equivalent to the divergenceless of the tensor $G^{\mu_{[p]}|\nu_{[q]}}$, that is, the latter satisfies the Noether identities

$$\partial^{\sigma_1} G_{\sigma_1 \ldots \sigma_{p+1} |\, \rho_1 \ldots \rho_{q+1}} = 0 = \partial^{\rho_1} G_{\sigma_1 \ldots \sigma_{p+1} |\, \rho_1 \ldots \rho_{q+1}} \,. \tag{2.6}$$

These identities are a direct consequence of the Bianchi ones (2.3). The Noether identities (2.6) ensure that the equations of motion can be written as

$$0 \approx G^{\mu_1 \ldots \mu_p |\, \nu_1 \ldots \nu_q} = \partial_\alpha H^{\alpha \mu_1 \ldots \mu_p |\, \nu_1 \ldots \nu_q} \,,$$

where

$$H^{\alpha \mu_1 \ldots \mu_p |}{}_{\nu_1 \ldots \nu_q} = \frac{1}{(p+1)!q!} \, \delta^{[\rho_1 \ldots \rho_q \alpha \mu_1 \ldots \mu_p]}_{[\nu_1 \ldots \nu_q \beta \sigma_1 \ldots \sigma_p]} \, \partial^{[\beta} \phi^{\sigma_1 \ldots \sigma_p]|}{}_{\rho_1 \ldots \rho_q} \,.$$

The symmetries of the tensor H correspond to the Young diagram $[p+1, q]$. This property will be useful in the computation of the local BRST cohomology.

2.3 Physical degrees of freedom

The "Ricci equation" (2.4) states that, on-shell, the field strength belongs to the irrep. $[p+1, q+1]$ of $O(D-1,1)$. The Bianchi identities together with (2.5) further imply that the on-shell non-vanishing components of the field strength belong to the unitary irrep. $[p, q]$ of the little group $O(D-2)$. Indeed, on-shell, gauge fields in the light-cone gauge are essentially field strengths [13], and the "Ricci equation" takes the form

$$\delta^{i_1 j_1} \phi_{i_1 \ldots i_p |\, j_1 \ldots j_q} \approx 0 \,.$$

where i and j denote light-cone indices ($i, j = 1, \ldots, D-2$). As a consistency check, one can note that the latter equation is non-trivial only when $p + q \geqslant D - 2$. The theory describes the correct physical degrees of freedom of a first-quantized massless particle propagating in flat space, *i.e.* , the latter particle provides a unitary irrep. of the group $IO(D-1,1)$.

We should stress that the exact analogue of all the previous properties hold for arbitrary mixed symmetry fields. This result was obtained by two of us and was mentioned in [7] but the detailed proof was not given there[3]. We take the opportunity to provide this extremely simple proof in Appendix A for the particular case of two-column gauge fields, since it already covers all the features of the general case for arbitrary mixed tensor gauge fields.

3 BRST construction

3.1 BRST deformation technique

Once one has a consistent free theory, it is natural to try to deform it into an interacting theory. The traditional Noether deformation procedure assumes that the deformed action

[3] The proof presented in this paper (Appendix A) provides an indirect proof that the light-cone gauge is reachable (so that the theory describes the correct number of physical degrees of freedom). We would like to underline the fact that the works [3, 5] assume (but do not contain any rigorous proof of) this fact. It would not be straightforward to prove it directly because the tower of ghosts is extremely complicated in the general case.

can be expressed as a power series in a coupling constant g, the zeroth-order term in the expansion describing the free theory S_0. The procedure is perturbative: one tries to construct the deformations order by order in the deformation parameter g.

Some physical requirements naturally come out:

- non-triviality: we reject *trivial* deformations arising from field-redefinitions that reduce to the identity at order g^0:
$$\phi \longrightarrow \phi' = \phi + g\,\theta(\phi, \partial\phi, \cdots) + \mathcal{O}(g^2)\,. \tag{3.7}$$

- consistency: a deformation of a theory is called *consistent* if the deformed theory possesses the same number of (possibly deformed) independent gauge symmetries, reducibility identities, *etc.*, as the system we started with. In other words, the number of physical degrees of freedom is unchanged.

- locality: The deformed action $S[\phi]$ must be a *local* functional. The deformation of the gauge transformations, *etc.*, must be local functions, as well as the field redefinitions.

We remind the reader that a local function of some set of fields φ^i is a smooth function of the fields φ^i and their derivatives $\partial\varphi^i$, $\partial^2\varphi^i$, ... up to some *finite* order, say k, in the number of derivatives. Such a set of variables φ^i, $\partial\varphi^i$, ..., $\partial^k\varphi^i$ will be collectively denoted by $[\varphi^i]$. Therefore, a local function of φ^i is denoted by $f([\varphi^i])$. A local p-form ($0 \leqslant p \leqslant D$) is a differential p-form the components of which are local functions:
$$\omega = \frac{1}{p!}\,\omega_{\mu_1\ldots\mu_p}(x, [\phi^i])\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\,.$$
A local functional is the integral of a local D-form.

As shown in [21], the Noether procedure can be reformulated in a BRST-cohomological formalism: the first-order non-trivial consistent local interactions are in one-to-one correspondence with elements of the cohomology $H^{D,0}(s|\,d)$ of the BRST differential s modulo the total derivative d, in maximum form-degree D and in ghost number 0. That is, one must compute the general solution of the cocycle condition
$$sa^{D,0} + db^{D-1,1} = 0, \tag{3.8}$$
where $a^{D,0}$ is a top-form of ghost number zero and $b^{D-1,1}$ a $(D-1)$-form of ghost number one, with the understanding that two solutions of (3.8) that differ by a trivial solution should be identified
$$a^{D,0} \sim a^{D,0} + sm^{D,-1} + dn^{D-1,0}$$
as they define the same interactions up to field redefinitions (3.7). The cocycles and coboundaries $a, b, m, n, \ldots$ are local forms of the field variables (including ghosts and antifields)

3.2 BRST spectrum

In the theories under consideration and according to the general rules of the BRST-antifield formalism, one associates with each gauge parameter $\alpha^{(i,j)}$ a ghost, and then to any field (including ghosts) a corresponding antifield (or antighost) of opposite Grassmann parity. More precisely, the spectrum of fields (including ghosts) and antifields is given by

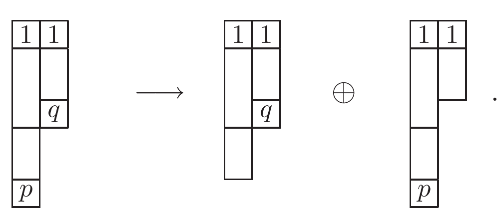
- the fields: $A^{(i,j)}_{\mu_{[p-i]}\,|\,\nu_{[q-j]}}$, where $A^{(0,0)}$ is identified with ϕ ;
- the antifields: $A^{*(i,j)\,\mu_{[p-i]}\,|\,\nu_{[q-j]}}$,

where $i = 0, ..., p-q$ and $j = 0, ..., q$. The symmetry properties of the fields $A^{(i,j)}_{\mu_{[p-i]}\,|\,\nu_{[q-j]}}$ and antifields $A^{*(i,j)\,\mu_{[p-i]}\,|\,\nu_{[q-j]}}$ are those of Young diagrams with two columns of lengths $p-i$ and $q-j$. To each field and antifield are associated a pureghost number and an antifield (or antighost) number. The pureghost number is given by $i+j$ for the fields $A^{(i,j)}$ and 0 for the antifields, while the antifield number is 0 for the fields and $i+j+1$ for the antifields $A^{*(i,j)}$. The Grassmann parity is given by the pureghost number (or the antighost number) modulo 2 . All this is summarized in Table 1.

	Young	*puregh*	*antigh*	Parity
$A^{(i,j)}$	$[p-i, q-j]$	$i+j$	0	$i+j$
$A^{*(i,j)}$	$[p-i, q-j]$	0	$i+j+1$	$i+j+1$

Table 1: *Symmetry, pureghost number, antighost number and parity of the (anti)fields.*

One can visualize the whole BRST spectrum in vanishing antighost number as well as the procedure that gives all the ghosts starting from $\phi_{\mu_{[p]}\,|\,\nu_{[q]}}$ on Figure 1, where the pureghost number increases from top down, by one unit at each line. At the top of Figure 1 lies the gauge field $\phi_{\mu_{[p]}\,|\,\nu_{[q]}}$ with pureghost number zero. At the level below, one finds the pureghost number one gauge parameters $A^{(1,0)}_{\mu_{[p-1]}\,|\,\nu_{[q]}}$ and $A^{(0,1)}_{\mu_{[p]}\,|\,\nu_{[q-1]}}$ whose respective symmetries are obtained by removing a box in the first (resp. second) column of the Young diagram $[p,q]$ corresponding to the gauge field $\phi_{\mu_{[p]}\,|\,\nu_{[q]}}$ (the rules that give the $(i+1)$-th generation ghosts from the i-th generation ones can be found in [4, 14]).



$\phi_{[p,q]}$ $\qquad$ $A^{(1,0)}_{[p-1,q]}$ $\qquad$ $A^{(0,1)}_{[p,q-1]}$

In pureghost number $p-q$, we obtain a set of ghosts containing $A^{(p-q,0)}_{\mu_{[q]}\nu_{[q]}} \sim [q,q]$. The Young diagram corresponding to the latter ghost is obtained by removing $p-q$ boxes from the first column of $[p,q]$.

If $q < p-q$, we do not have to reach the pureghost level $p-q$ to find the p-form ghost $A^{(0,q)}_{\mu_{[p]}} \sim [p\,,0]$. If $2q \geqslant p$, we have to remove additional boxes from the second column

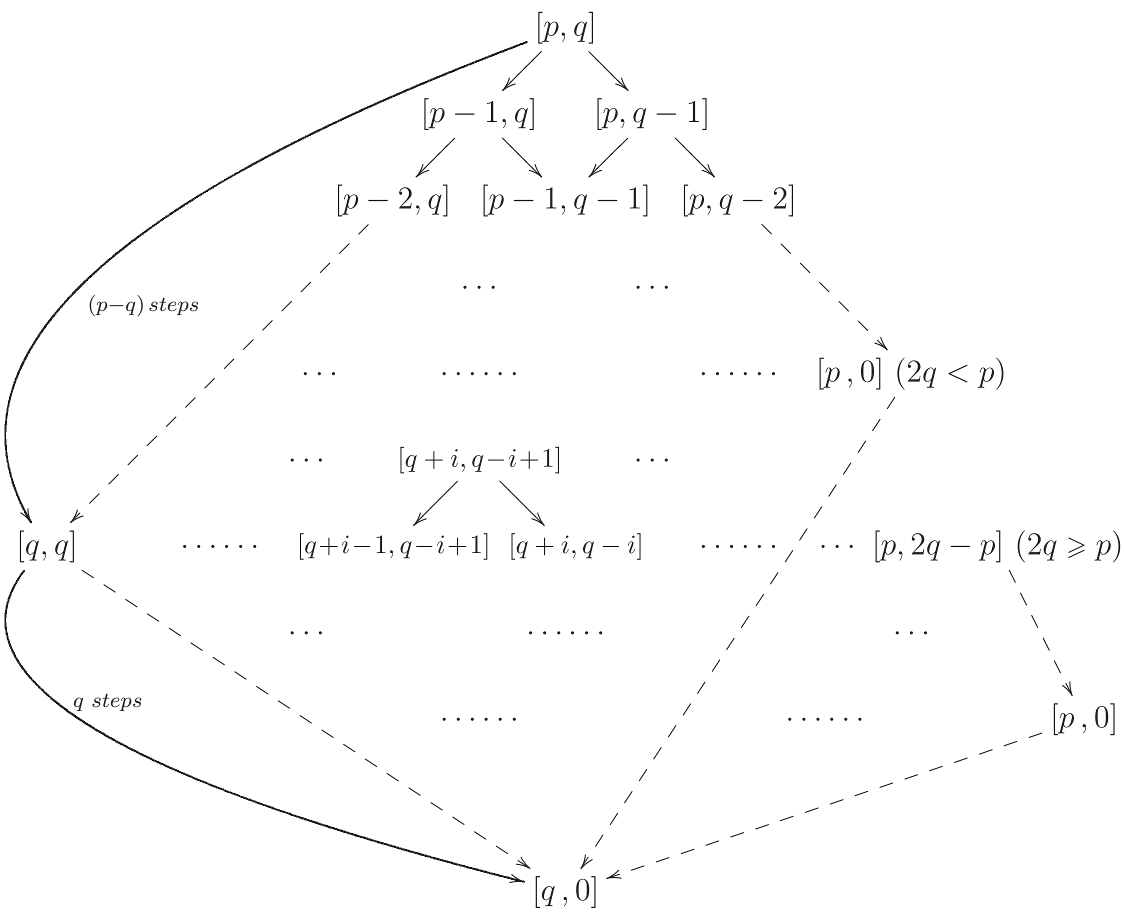


Figure 1: *Antighost-zero BRST spectrum of $[p, q]$−type gauge field.*

of $[p, q]$ in order to empty it completely and obtain the p-form ghost $A^{(0,q)}_{\mu_{[p]}}$. The Young diagrams of the remaining ghosts are obtained by further removing boxes from the Young diagram corresponding to the ghost $A^{(0,q)}_{\mu_{[p]}}$ with $puregh = q$. This procedure will terminate at pureghost number p with the q-form ghost $A^{(p-q,q)}_{\mu_{[q]}} \sim [q, 0]$. It is not possible to find ghosts $A_{\mu_{[r]}|\nu_{[s]}}$ with $r, s < q$, since it would mean that two boxes from a same row would have been removed from $[p, q]$, which is not allowed [4, 14].

The antighost sector has exactly the same structure as the ghost sector of Figure 1, where each ghost $A^{(i,j)}$ is replaced by its antighost $A^{*(i,j)}$.

3.3 BRST differential

The BRST differential s of the free theory (2.1), (2.2) is generated by the functional

$$W_0 = S_0[\phi] + \int d^Dx \quad \Big[\sum_{i=0}^{p-q} \sum_{j=0}^{q} (-)^{i+j} A^{*(i,j)\,\mu_1\ldots\mu_{p-i}|\,\nu_1\ldots\nu_{q-j}} \times \big(\partial_{[\mu_1} A^{(i+1,j)}_{\mu_2\ldots\mu_{p-i}]|\,\nu_1\ldots\nu_{q-j}} - b_{i+1,j}\, A^{(i,j+1)}_{\mu_1\ldots\mu_{p-i}|\,[\nu_1\ldots\nu_{q-j-1},\nu_{q-j}]}\big)\Big],$$

with the convention that $A^{(p-q+1,j)} = A^{(i,q+1)} = A^{*(-1,j)} = A^{*(i,-1)} = 0$. More precisely, W_0 is the generator of the BRST differential s of the free theory through

$$sA = (W_0, A)_{a.b.} \,,$$

where the antibracket $(\ ,\ )_{a.b.}$ is defined by $(A,B)_{a.b.} = \frac{\delta^R A}{\delta \Phi^I}\frac{\delta^L B}{\delta \Phi^*_I} - \frac{\delta^R A}{\delta \Phi^*_I}\frac{\delta^L B}{\delta \Phi^I}$. The functional W_0 is a solution of the *master equation*

$$(W_0, W_0)_{a.b.} = 0 \,.$$

The BRST-differential s decomposes into $s = \gamma + \delta$. The first piece γ , the differential along the gauge orbits, increases the pureghost number by one unit, whereas the Koszul-Tate differential δ decreases the antighost (or antifield) number by one unit. A $\mathbb{Z}$-grading called *ghost number* (or gh) corresponds to the differential s . We have

$$gh = puregh - antigh \,.$$

The action of γ and δ on the BRST variables is zero, except

$$\begin{aligned}
\gamma A^{(i,j)}_{\mu_{[p-i]}|\,\nu_{[q-j]}} &= \partial_{[\mu_1} A^{(i+1,j)}_{\mu_2\ldots\mu_{p-i}]|\,\nu_{[q-j]}} \\
&\quad + b_{i,j}\left(A^{(i,j+1)}_{\mu_{[p-i]}|\,[\nu_{[q-j-1]},\nu_{q-j}]} + a_{i,j} A^{(i,j+1)}_{\nu_{[q-j]}[\mu_{q-j+1}\ldots\mu_{p-i}|\,\mu_{[q-j-1]},\mu_{q-j}]}\right) \\
\delta A^{*(0,0)\ \mu_{[p]}|\,\nu_{[q]}} &= G^{\mu_{[p]}|\,\nu_{[q]}} \\
\delta A^{*(i,j)\ \mu_{[p-i]}|\,\nu_{[q-j]}} &= (-)^{i+j}\left(\partial_\sigma A^{*(i-1,j)\ \sigma\mu_{[p-i]}|\,\nu_{[q-j]}} - \frac{1}{p-i+1}\,\partial_\sigma A^{*(i-1,j)\ \nu_1\mu_{[p-i]}|\,\sigma\nu_2\ldots\nu_{q-j}}\right) \\
&\quad +(-)^{i+j+1} b_{i+1,j-1}\partial_\sigma A^{*(i,j-1)\ \mu_{[p-i]}|\,\nu_{[q-j]}\sigma} \,,
\end{aligned}$$

where the last equation holds only for (i,j) different from $(0,0)$.

For later computations, it is useful to define a unique antifield for each antighost number:

$$C^{*\ \mu_1\ldots\mu_q|\,\nu_1\ldots\nu_j}_{p+1-j} = \sum_{k=0}^{j} \epsilon_{k,j} A^{*(p-q-j+k,q-k)\ \mu_1\ldots\mu_q[\nu_{k+1}\ldots\nu_j|\,\nu_1\ldots\nu_k]}$$

for $0 \leqslant j \leqslant p$, and, in antighost zero, the following specific combination of single derivatives of the field

$$C^{*\ \mu_1\ldots\mu_q|\,\nu_1\ldots\nu_{p+1}}_0 = \epsilon_{q,p+1} H^{\mu_1\ldots\mu_q[\nu_{q+1}\ldots\nu_{p+1}|\,\nu_1\ldots\nu_q]} \,,$$

where $\epsilon_{k,j}$ vanishes for $k > q$ and for $j - k > p - q$, and is given in the other cases by:

$$\epsilon_{k,j} = (-)^{pk+j(k+p+q)+\frac{k(k+1)}{2}} \frac{\binom{k}{p+1}\binom{k}{j}}{\binom{k}{q}}$$

where $\binom{m}{n}$ are the binomial coefficients $(n \geqslant m)$. Some properties of the new variables C^*_k are summarized in Table 2.

	Young diagram	*puregh*	*antigh*	Parity
C_k^*	$[q]\otimes[p+1-k]-[p+1]\otimes[q-k]$	0	k	k

Table 2: *Young representation, pureghost number, antighost number and parity of the antifields* C_k^*.

The symmetry properties of C_k^* are denoted by

$$[q]\otimes[p+1-k]\,-\,[p+1]\otimes[q-k]$$

which means that they have the symmetry properties corresponding to the tensor product of a column $[q]$ by a column $[p+1-k]$ from which one should substract (when $k\leqslant q$) all the Young diagrams appearing in the tensor product $[p+1]\otimes[q-k]$.

The antifields $C_k^{*\,\mu_{[q]}|\,\nu_{[p+1-k]}}$ have been defined in order to obey the following relations:

$$\begin{aligned}\delta C^{*\,\mu_1\ldots\mu_q|\,\nu_1\ldots\nu_j}_{p+1-j} &= \partial_\sigma C^{*\,\mu_1\ldots\mu_q|\,|\,\nu_1\ldots\nu_j\sigma}_{p-j} \quad \text{for} \quad 0\leqslant j\leqslant p\,,\\ \delta C^{*\,\mu_1\ldots\mu_q|\,\nu_1\ldots\nu_{p+1}}_0 &= 0\,.\end{aligned} \tag{3.9}$$

If we further define the inhomogeneous form

$$\tilde H^{\mu_1\ldots\mu_q}\equiv\sum_{j=0}^{p+1}C^{*\,D-j\;\mu_1\ldots\mu_q}_{p+1-j}\;,$$

where

$$C^{*\,D-j\;\mu_1\ldots\mu_q}_{p+1-j}\equiv(-)^{jp+\frac{j(j+1)}{2}}\frac{1}{j!(D-j)!}\,C^{*\,\mu_1\ldots\mu_q|\,\nu_1\ldots\nu_j}_{p+1-j}\,\epsilon_{\nu_1\ldots\nu_D}dx^{\nu_{j+1}}\ldots dx^{\nu_D}\;,$$

then, as a consequence of (3.9), any polynomial $P(\tilde H)$ in $\tilde H^{\mu_1\ldots\mu_q}$ will satisfy

$$(\delta+d)P(\tilde H)=0\,. \tag{3.10}$$

The polynomial $\tilde H$ is not invariant under gauge transformations. It is therefore useful to introduce another polynomial, $\tilde{\mathcal H}$, with an explicit x-dependance, that *is* invariant. $\tilde{\mathcal H}$ is defined by

$$\tilde{\mathcal H}_{\mu_{[q]}}\equiv\sum_{j=1}^{p+1}C^{*\,D-p-1+j}_{j\;\mu_{[q]}}+\tilde a\,\epsilon_{[\mu_{[q]}\sigma_{[p+1]}\tau_{[D-p-q-1]}]}K^{q+1\,\sigma_{[p+1]}}x^{\tau_1}dx^{\tau_2}\ldots dx^{\tau_{D-p-q-1}}\;,$$

where $\tilde a=(-)^{\frac{p(p-1)+q(q-1)}{2}}\frac{1}{q!q!(p+q+1)!(p+1-q)!(D-p-q-1)!}$. One can check that $\tilde{\mathcal H}=\tilde H+dm_0^{D-p-2}$. This fact has the consequence that polynomials in $\tilde{\mathcal H}$ also satisfy $(\delta+d)P(\tilde{\mathcal H})=0$.

4 Cohomology of γ

We hereafter give the content of $H(\gamma)$. Subsequently, we explain the procedure that we followed in order to obtain that result.

Theorem 4.1. *The cohomology of γ is isomorphic to the space of functions depending on*

- *the antifields and their derivatives $[A^{*(i,j)}]$,*
- *the curvature and its derivatives $[K]$,*
- *the p-th generation ghost $A^{(p-q,q)}$ and*
- *the curl $D^0_{\mu_1\ldots\mu_{p+1}} \equiv (-)^q \partial_{[\mu_1} A^{(0,q)}_{\mu_2\ldots\mu_{p+1}]}$ of the q-th generation ghost $A^{(0,q)}$.*

$$H(\gamma) \simeq \left\{ f\left([A^{*(i,j)}], [K], A^{(p-q,q)}, D^0_{\mu_1\ldots\mu_{p+1}} \right) \right\} .$$

Proof : The antifields and all their derivatives are annihilated by γ. Since they carry no pureghost degree by definition, they cannot be equal to the γ-variation of any quantity. Hence, they obviously belong to the cohomology of γ.

To compute the γ-cohomology in the sector of the field, the ghosts and all their derivatives, we split the variables into three sets of *independent* variables obeying respectively $\gamma u^\ell = v^\ell$, $\gamma v^\ell = 0$ and $\gamma w^i = 0$. The variables u^ℓ and v^ℓ form so-called "contractible pairs" and the cohomology of γ is therefore generated by the variables w^i (see e.g. [28], Theorem 8.2).

We decompose the spaces spanned by the derivatives $\partial_{\mu_1\ldots\mu_k} A^{(i,j)}$, $k \geqslant 0$, $0 \leqslant i \leqslant p-q$, $0 \leqslant j \leqslant q$, into irreps of $GL(D,\mathbb{R})$ and use the structure of the reducibility conditions (see Figures 2. and 3.) in order to group the variables into contractible pairs.

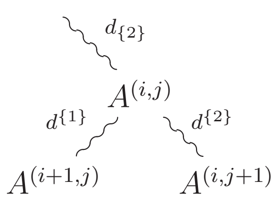


Figure 2

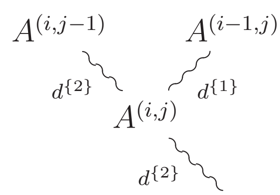
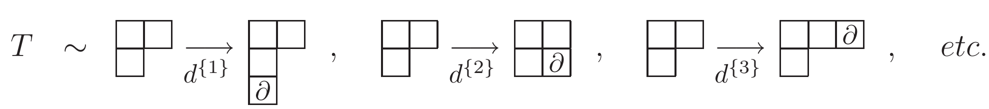


Figure 3

We use the differential operators $d^{\{i\}}$, $i = 1, 2, \ldots$ (see [4] for a general definition) which act, for instance on Young-symmetry type tensor fields $T_{[2,1]}$, as follows:



For fixed i and j the set of ghosts $A^{(i,j)}$ and all their derivatives decompose into three types of independent variables:

$$[A^{(i,j)}] \quad \longleftrightarrow \quad \mathcal{O}A^{(i,j+1)}, \mathcal{O}d^{\{1\}}A^{(i,j+1)}, \mathcal{O}d^{\{2\}}A^{(i,j+1)}, \mathcal{O}d^{\{1\}}d^{\{2\}}A^{(i,j+1)}$$

where $\mathcal{O}$ denotes any operator of the type $\prod_{m\geqslant 3} d^{\{m\}}$ or the identity.

Different cases arise depending on the position of the field $A^{(i,j)}$ in Figure 1. We have to consider fields that sit in the interior, on a border or at a corner of the diagram.

- Interior

  In this case, all the ghosts $A^{(i,j)}$ and their derivatives form u^ℓ or v^ℓ variables. Indeed, we have the relations

$$\begin{array}{rcl} \gamma A^{(i,j)} & \propto & [d^{\{1\}}A^{(i+1,j)} + d^{\{2\}}A^{(i,j+1)}]\,, \\ \gamma[d^{\{1\}}A^{(i+1,j)} - d^{\{2\}}A^{(i,j+1)}] & = & 0\,, \\ \gamma[d^{\{1\}}A^{(i+1,j)} + d^{\{2\}}A^{(i,j+1)}] & \propto & d^{\{1\}}d^{\{2\}}A^{(i,j+1)}\,, \\ \gamma[d^{\{1\}}d^{\{2\}}A^{(i,j+1)}] & = & 0\,, \end{array}$$

  and $\mathcal{O}$ commutes with γ. From which we conclude that one can perform a change of variable from the sets $[A^{(i,j)}]$ to the contractible pairs

$$\begin{array}{rcl} u^\ell & \leftrightarrow & \mathcal{O}A^{(i,j+1)}\,,\, \mathcal{O}[d^{\{1\}}A^{(i,j+1)} + d^{\{2\}}A^{(i,j+1)}] \\ v^\ell & \leftrightarrow & \mathcal{O}[d^{\{1\}}A^{(i,j+1)} - d^{\{2\}}A^{(i,j+1)}]\,,\, \mathcal{O}d^{\{1\}}d^{\{2\}}A^{(i,j+1)} \end{array}$$

  so that the ghosts $A^{(i,j)}$ in the interior and all their derivatives do not appear in $H(\gamma)$.

- Lower corner

  On the one hand, we have $\gamma A^{(p-q,q)}_{[q,0]} = 0$. As the operator γ introduces a derivative, $A^{(p-q,q)}_{[q,0]}$ cannot be γ-exact. As a result, $A^{(p-q,q)}_{[q,0]}$ is a w^i-variable and thence belongs to $H(\gamma)$. On the other hand, we find $\partial_\nu A^{(p-q,q)}_{\mu_1\ldots\mu_q} = \gamma[A^{(p-q-1,q)}_{\nu\mu_1\ldots\mu_q} + (-)^{p-q}\frac{q}{p+1}A^{(p-q,q-1)}_{\mu_1\ldots\mu_q|\,\nu}]$, which implies that all the derivatives of $A^{(p-q,q)}$ do not appear in $H(\gamma)$.

- Border

  If a ghost $A^{(i,j)}$ stands on a border of Figure 1, it means that either (i) its reducibility relation involves only one ghost (see e.g. Fig. 3), or (ii) there exists only one field whose reducibility relation involves $A^{(i,j)}$ (see e.g. Fig. 2):

  (i) Suppose $A^{(i,j)}$ stands on the left-hand (lower) edge of Figure 1. We have the relations

$$\begin{array}{rcl} \gamma A^{(i,j)} & \propto & d^{\{2\}}A^{(i,j+1)}\,, \\ \gamma[d^{\{2\}}A^{(i,j+1)}] & = & 0\,, \\ \gamma[d^{\{1\}}A^{(i,j)}] & \propto & d^{\{1\}}d^{\{2\}}A^{(i,j+1)}\,, \\ \gamma[d^{\{1\}}d^{\{2\}}A^{(i,j+1)}] & = & 0\,, \end{array}$$

  so that the corresponding sets $[A^{(i,j)}]$ on the left-hand edge do not contribute to $H(\gamma)$. We reach similar conclusion if $A^{(i,j)}$ lies on the right-hand (higher) border of Figure 1, substituting $d^{\{1\}}$ for $d^{\{2\}}$ when necessary.

(ii) Since, by assumption, $A^{(i,j)}$ does not sit in a corner of Fig. 1 (but on the higher left-hand or lower right-hand border), its reducibility transformation involves two ghosts, and we proceed as if it were in the interior. The only difference is that $\mathcal{O}d^{\{1\}}d^{\{2\}}A^{(i,j)}$ will be equal to either $\gamma\mathcal{O}d^{\{1\}}A^{(i,j-1)}$ or $\gamma\mathcal{O}d^{\{2\}}A^{(i-1,j)}$, depending whether the field above $A^{(i,j)}$ is $A^{(i-1,j)}$ or $A^{(i,j-1)}$.

- <u>Left-hand corner</u>

  In this case, the ghost $A^{(i,j)}$ is characterized by a squared-shape Young diagram (it is the only one with this property). Its reducibility transformation involves only one ghost and there exists only one field whose reducibility transformation involves $A^{(i,j)}$. Because of its symmetry properties, $d^{\{2\}}A^{(i,j)} \sim d^{\{1\}}A^{(i,j)}$. Better, $d^{\{2\}}$ is not well-defined on $A^{(i,j)}$, it is only well-defined on $d^{\{1\}}A^{(i,j)}$. Therefore, the derivatives $\partial_{\mu_1\dots\mu_k}A^{(i,j)}$ decompose into $\mathcal{O}A^{(i,j)}$, $\mathcal{O}d^{\{1\}}A^{(i,j)}$ and $\mathcal{O}d^{\{1\}}d^{\{2\}}A^{(i,j)}$. The first set $\mathcal{O}A^{(i,j)}$ form u^ℓ-variables associated with $\mathcal{O}d^{\{2\}}A^{(i,j+1)}$. The second set is grouped with $\mathcal{O}d^{\{1\}}d^{\{2\}}A^{(i,j+1)}$, and the third one form v^ℓ-variables with $\mathcal{O}d^{\{2\}}A^{(i-1,j)}$.

- <u>Upper-corner</u>

  In the case where $A^{(i,j)}$ is the gauge field, we proceed exactly as in the "Interior" case, except that the variables $\mathcal{O}d^{\{1\}}d^{\{2\}}A^{(i,j)} = 0$ are not grouped with any other variables any longer. They constitute true w^i-variables and are thus present in $H(\gamma)$. Recalling the definition of the curvature K, we have $\mathcal{O}d^{\{1\}}d^{\{2\}}A^{(i,j)} \propto [K]$.

- <u>Right-hand corner</u>

  In this case, the field $A^{(i,j)}$ is the p-form ghost $A^{(0,q)}_{[p]}$. We have the (u,v)-pairs $(\mathcal{O}d^{\{2\}}A^{(0,q)}, \mathcal{O}d^{\{1\}}d^{\{2\}}A^{(1,q)})$, $(\mathcal{O}d^{\{1\}}A^{(0,q-1)}, \mathcal{O}d^{\{1\}}d^{\{2\}}A^{(0,q)})$.

  The derivative $d^{\{1\}}A^{(0,q)}_{[p]} \propto D^0_{[p+1]}$ is a w^i-variable since it is invariant and no other variable $\partial_{\mu_1\dots\mu_k}A^{(i,j)}$ possesses the same symmetry.

□

In the sequel, the polynomials $\alpha([K],[A^*])$ in the curvature, the antifields and all their derivatives will be called "invariant polynomials". We will denote by $\mathcal{N}$ the algebra generated by all the ghosts and the non-invariant derivatives of the field ϕ. The entire algebra of the fields and antifields is then generated by the invariant polynomials and the elements of $\mathcal{N}$.

5 Invariant Poincaré lemma

The space of *invariant* local forms is the space of (local) forms that belong to $H(\gamma)$. The algebraic Poincaré lemma tells us that any closed form is exact. However, if the form is furthermore invariant, it is not guaranteed that the form is exact in the space of invariant forms. The following lemma tells us more about this important subtlety, in a limited range of form degree.

Lemma 5.1 (Invariant Poincaré lemma in form degree $k < p+1$). *Let α^k be an invariant local k-form, $k < p+1$.*

$$\textit{If} \quad d\alpha^k = 0\,, \quad \textit{then} \quad \alpha^k = Q(K^{q+1}_{\mu_1\dots\mu_{p+1}}) + d\beta^{k-1}\,,$$

where Q is a polynomial in the $(q+1)$-forms

$$K^{q+1}_{\mu_1\dots\mu_{p+1}} \equiv K_{\mu_1\dots\mu_{p+1}|\,\nu_1\dots\nu_{q+1}} dx^{\nu_1}\dots dx^{\nu_{q+1}}\,,$$

while β^{k-1} is an invariant local form.

A closed invariant local form of form-degree $k < p+1$ and of strictly positive antighost number is always exact in the space of invariant local forms.

The proof is directly inspired from the one given in [26] (Theorem 6).

5.1 Beginning of the proof of the invariant Poincaré lemma

The second statement of the lemma (*i.e.* the case $antigh(\alpha^k) \neq 0$) is part of a general theorem (see e.g. [23]) which holds without any restriction on the form-degree. It will not be reviewed here.

We will thus assume that $antigh(\alpha^k) = 0$, and prove the first part of Lemma 5.1 by induction:

Induction basis: For $k = 0$, the invariant Poincaré lemma is trivially satisfied: $d\alpha^0 = 0$ implies that α^0 is a constant by the usual Poincaré lemma.

Induction hypothesis: The lemma holds in form degree k' such that $0 \leqslant k' < k$

Induction step: We will prove in the sequel that under the induction hypothesis, the lemma holds in form degree k.

Because $d\alpha^k = 0$ and $\gamma\alpha^k = 0$, we can build a descent as follows

$$\begin{aligned} d\alpha^k = 0 \Rightarrow \alpha^k &= da^{k-1,0} && (5.11)\\ 0 &= \gamma a^{k-1,0} + da^{k-2,1} && (5.12)\\ &\vdots \\ 0 &= \gamma a^{k-j,j-1} + da^{k-j-1,j} && (5.13)\\ 0 &= \gamma a^{k-j-1,j}\,, && (5.14) \end{aligned}$$

where $a^{r,i}$ is a r-form of pureghost number i . The pureghost number of $a^{r,i}$ must obey $0 \leqslant i \leqslant k-1$. Of course, since we assume $k < p+1$, we have $i < p$. The descent stops at (5.14) either because $k-j-1 = 0$ or because $a^{k-j-1,j}$ is invariant. The case $j = 0$ is trivial since it gives immediately $\alpha^k = d\beta^{k-1}$, where $\beta^{k-1} \equiv a^{k-1,0}$ is invariant. Accordingly, we assume from now on that $j > 0$.

Since we are dealing with a descent, it is helpful to introduce one of its building blocks, which is the purpose of the next subsection. We will complete the induction step in Subsection 5.3.

5.2 A descent of γ modulo d

Let us define the following differential forms built up from the ghosts

$$D^l_{\mu_1\dots\mu_{p+1}} \equiv (-)^{l(q+1)+q}\partial_{[\mu_1}A^{(0,q-l)}_{\mu_2\dots\mu_{p+1}]|\,\nu_1\dots\nu_l}dx^{\nu_1}\dots dx^{\nu_l}\,,$$

for $0\leqslant l\leqslant q$. It is easy to show that these fields verify the following descent:

$$\begin{aligned}\gamma(D^0_{\mu_1\dots\mu_{p+1}}) &= 0\,, &&(5.15)\\ \gamma(D^{l+1}_{\mu_1\dots\mu_{p+1}})+dD^l_{\mu_1\dots\mu_{p+1}} &= 0\,, \qquad 0\leqslant l\leqslant q-1\,,\\ dD^q_{\mu_1\dots\mu_{p+1}} &= K^{q+1}_{\mu_1\dots\mu_{p+1}}\,. &&(5.16)\end{aligned}$$

It is convenient to introduce the inhomogeneous form

$$D_{\mu_1\dots\mu_{p+1}}=\sum_{l=0}^{q}D^l_{\mu_1\dots\mu_{p+1}}$$

because it satisfies a so-called "Russian formula"

$$(\gamma+d)D_{\mu_1\dots\mu_{p+1}}=K^{q+1}_{\mu_1\dots\mu_{p+1}}\,, \tag{5.17}$$

which is a compact way of writing the descent (5.15)–(5.16).

Let $\omega_{(n,m)}$ be a homogeneous polynomial of degree m in D and of degree n in K. Its decomposition is

$$\omega_{(n,m)}(K,D)=\omega^{n(q+1)+mq,0}+\dots+\omega^{n(q+1)+j,mq-j}+\dots+\omega^{n(q+1),mq}$$

where $\omega^{n(q+1)+j,mq-j}$ has form degree $n(q+1)+j$ and pureghost number $mq-j$. Due to (5.17), the polynomial satisfies

$$(\gamma+d)\omega_{(n,m)}=K^{q+1}_{\mu_1\dots\mu_{p+1}}\frac{\partial^L\omega_{(n,m)}}{\partial D_{\mu_1\dots\mu_{p+1}}}\,, \tag{5.18}$$

the form degree decomposition of which leads to the descent

$$\begin{aligned}\gamma(\omega^{n(q+1),mq}) &= 0\,,\\ \gamma(\omega^{n(q+1)+j+1,mq-j-1})+d\omega^{n(q+1)+j,mq-j} &= 0\,,\quad 0\leqslant j\leqslant q-1\\ \gamma(\omega^{n(q+1)+q+1,(m-1)q-1})+\,d\omega^{n(q+1)+q,(m-1)q} &= K^{q+1}_{\mu_1\dots\mu_{p+1}}\Big[\frac{\partial^L\omega}{\partial D_{\mu_1\dots\mu_{p+1}}}\Big]^{n(q+1),(m-1)q} &&(5.19)\end{aligned}$$

where $[\frac{\partial\omega}{\partial D}]^{n(q+1),(m-1)q}$ denotes the component of form degree $n(q+1)$ and pureghost equal to $(m-1)q$ of the derivative $\frac{\partial\omega}{\partial D}$. This component is the homogeneous polynomial of degree $m-1$ in the variable D^0,

$$\Big[\frac{\partial\omega}{\partial D_{\mu_1\dots\mu_{p+1}}}\Big]^{n(q+1),(m-1)q}=\frac{\partial\omega}{\partial D_{\mu_1\dots\mu_{p+1}}}|_{D=D^0}\,.$$

The right-hand-side of (5.19) vanishes if and only if the right-hand-side of (5.18) does.

Two cases arise depending on whether the r.h.s. of (5.18) vanishes or not.

- The r.h.s. of (5.18) vanishes: then the descent is said not to be obstructed in any strictly positive pureghost number and goes all the way down to the bottom equations

$$\begin{aligned} \gamma(\omega^{n(q+1)+mq,0}) + d\omega^{n(q+1)+mq+1,1} &= 0\,, \quad 0 \leqslant j \leqslant q-1 \\ d(\omega^{n(q+1)+mq,0}) &= 0\,. \end{aligned}$$

- The r.h.s. of (5.18) is not zero : then the descent is obstructed after q steps. It is not possible to find an $\tilde{\omega}^{n(q+1)+q+1,(m-1)q-1}$ such that

$$\gamma(\tilde{\omega}^{n(q+1)+q+1,(m-1)q-1}) + \, d\omega^{n(q+1)+q,(m-1)q} = 0\,,$$

  because the r.h.s. of (5.19) is an element of $H(\gamma)$. This element is called the *obstruction* to the descent. One also says that this obstruction cannot be lifted more than q times, and $\omega^{n(q+1),mq}$ is the top of the ladder (in this case it must be an element of $H(\gamma)$).

This covers the general type of ladder (descent as well as lift) that do not contain the p-th generation ghost $A^{(p-q,q)}$.

5.3 End of the proof of the invariant Poincaré lemma

As $j < p$, Theorem 4.1 implies that the equation (5.14) has non-trivial solutions only when $j = mq$ for some integer m

$$a^{k-mq-1,mq} = \sum_I \alpha_I^{k-mq-1}\,\omega_I^{0,mq}\,, \tag{5.20}$$

up to some γ-exact term. The α_I^{k-mq-1}'s are invariant forms, and $\{\omega_I^{0,mq}\}$ is a basis of polynomials of degree m in the variable D^0. The ghost $A^{(p-q,q)}$ are absent since the pureghost number is $j = mq < p$.

The equation (5.13) implies $d\alpha_I^{k-mq-1} = 0$. Together with the induction hypothesis, this implies

$$\alpha_I^{k-mq-1} = P_I(K^{q+1}_{\mu_1\ldots\mu_{p+1}}) + d\beta^{k-j-2} \quad , \tag{5.21}$$

where the polynomials P_I of order n are present iff $k - mq - 1 = n(q+1)$. Inserting (5.21) into (5.20) we find that, up to trivial redefinitions, $a^{k-j-1,j}$ is a polynomial in $K^{q+1}_{\mu_1\ldots\mu_{p+1}}$ and $D^0_{\mu_1\ldots\mu_{p+1}}$.

From the analysis performed in Subsection 5.2, we know that such an $a^{k-j-1,j}$ can be lifted at most q times. Therefore, $a^{k-j-1,j}$ belongs to a descent of type (5.11)–(5.14) only if $j = q$. Without loss of generality we can thus take $a_q^{k-q-1} = P(K^{q+1}_{\mu_1\ldots\mu_{p+1}}, D^0)$ where P is a homogeneous polynomial with a linear dependence in D^0 (since $m = 1$). In such a case, it can be lifted up to (5.11). Furthermore, because $a^{k-1,0}$ is defined up to an invariant form $\beta^{k-1,0}$ by the equation (5.12), the term $da^{k-1,0}$ of (5.11) must be equal to the sum

$$da^{k-1,0} = \underbrace{P(K^{q+1}, K^{q+1})}_{\equiv Q(K^{q+1}_{\mu_1\ldots\mu_{p+1}})} + d\beta^{k-1,0}$$

of a homogeneous polynomial Q in K^{q+1} (the lift of the bottom) and a form d-exact in the invariants. ☐

6 Cohomology of δ modulo d: $H^D_k(\delta|\, d)$

In this section, we compute the cohomology of δ modulo d in top form-degree and antighost number k, for $k \geqslant q$. We will also restrict ourselves to $k > 1$. The group $H^D_1(\delta|\, d)$ describes the infinitely many conserved currents and will not be studied here.

Let us first recall a general theorem (Theorem 9.1 in [24]).

Theorem 6.1. *For a linear gauge theory of reducibility order $p-1$,*

$$H^D_k(\delta|\, d) = 0 \; for \; k > p+1\,.$$

The computation of the cohomology groups $H^D_k(\delta|\, d)$ for $q \leqslant k \leqslant p+1$ follows closely the procedure used for p-forms in [26]. It relies on the following theorems:

Theorem 6.2. *Any solution of $\delta a^D + db^{D-1} = 0$ that is at least bilinear in the antifields is necessarily trivial.*

The proof of Theorem 6.2 is similar to the proof of Theorem 11.2 in [24] and will not be repeated here.

Theorem 6.3. *A complete set of representatives of $H^D_{p+1}(\delta|\, d)$ is given by the antifields $C^{*\,D}_{p+1\,\mu_1\ldots\mu_q}$, i.e.*

$$\delta a^D_{p+1} + da^{D-1}_p = 0 \;\Rightarrow\; a^D_{p+1} = \lambda^{\mu_{[q]}} C^{*\,D}_{p+1\,\mu_{[q]}} + \delta b^D_{p+2} + db^{D-1}_{p+1}\,,$$

where the $\lambda^{[\mu_1\ldots\mu_q]}$ are constants.

Proof: Candidates: any polynomial of antighost number $p+1$ can be written

$$a^D_{p+1} = \Lambda^{[\mu_1\ldots\mu_q]} C^{*\,D}_{p+1\,[\mu_1\ldots\mu_q]} + \mu^D_{p+1} + \delta b^D_{p+2} + db^{D-1}_{p+1}\,,$$

where Λ does not involve the antifields and where μ^D_{p+1} is at least quadratic in the antifields. The cocycle condition $\delta a^D_{p+1} + da^{D-1}_p = 0$ then implies

$$-\Lambda^{[\mu_1\ldots\mu_q]} dC^{*\,D-1}_{p\,[\mu_1\ldots\mu_q]} + \delta(\mu^D_{p+1} + db^{D-1}_{p+1}) = 0\,.$$

By taking the Euler-Lagrange derivative of this equation with respect to $C^*_{p\,[\mu_1\ldots\mu_q]|\,\nu}$, one gets the weak equation $\partial^\nu \Lambda^{[\mu_1\ldots\mu_q]} \approx 0$. Considering ν as a form index, one sees that Λ belongs to $H^0_0(d|\,\delta)$. The isomorphism $H^0_0(d|\,\delta)/\mathbb{R} \cong H^D_D(\delta|\, d)$ (see [24]) combined with the knowledge of $H^D_D(\delta|\, d) \cong 0$ (by Theorem 6.1) implies $\Lambda^{[\mu_1\ldots\mu_q]} = \lambda^{[\mu_1\ldots\mu_q]} + \delta\nu_1^{[\mu_1\ldots\mu_q]}$ where $\lambda^{[\mu_1\ldots\mu_q]}$ is a constant. The term $\delta\nu_1^{[\mu_1\ldots\mu_q]} C^{*\,D}_{p+1\,[\mu_1\ldots\mu_q]}$ can be rewritten as a term at least bilinear in the antifields up to a δ-exact term. Inserting $a^D_{p+1} = \lambda^{[\mu_1\ldots\mu_q]} C^{*\,D}_{p+1\,\mu_1\ldots\mu_q} + \mu^D_{p+1} + \delta b^D_{p+2} + db^{D-1}_{p+1}$ into the cocycle condition, we see that μ^D_{p+1} has to be a solution of $\delta\mu^D_{p+1} + db^{D-1} = 0$ and is therefore trivial by Theorem 6.2.

Non-triviality: It remains to show that the cocycles $a^D_{p+1} = \lambda C^{*\,D}_{p+1}$ are non-trivial. Indeed one can prove that $\lambda C^{*\,D}_{p+1} = \delta u^D_{p+2} + dv^{D-1}_{p+1}$ implies that $\lambda C^{*\,D}_{p+1}$ vanishes. It

is straightforward when u^D_{p+2} and v^{D-1}_{p+1} do not depend explicitly on x: δ and d bring in a derivative while $\lambda C^{*\,D}_{p+1}$ does not contain any. If u and v depend explicitly on x, one must expand them and the equation $\lambda C^{*\,D}_{p+1} = \delta u^D_{p+2} + dv^{D-1}_{p+1}$ according to the number of derivatives of the fields and antifields to reach the conclusion. Explicitly, $u^D_{p+2} = u^D_{p+2,\,0} + \ldots + u^D_{p+2,\,l}$ and $v^{D-1}_{p+1} = v^{D-1}_{p+1,\,0} + \ldots + v^{D-1}_{p+1,\,n}$. If $n > l$, the equation in degree $n+1$ reads $0 = d'v^{D-1}_{p+1,\,n}$ where d' does not differentiate with respect to the explicit dependence in x. This in turn implies that $v^{D-1}_{p+1,\,n} = d'\tilde{v}^{D-1}_{p+1,\,n-1}$ and can be removed by redefining v^{D-1}_{p+1}: $v^{D-1}_{p+1} \to v^{D-1}_{p+1} - d\tilde{v}^{D-1}_{p+1,\,n-1}$. If $l > n$, the equation in degree $l+1$ is $0 = \delta u^D_{p+2,\,l}$ and implies, together with the acyclicity of δ, that one can remove $u^D_{p+2,\,l}$ by a trivial redefinition of u^D_{p+2}. If $l = n > 0$, the equation in degree $l+1$ reads $0 = \delta u^D_{p+2,\,l} + d'v^{D-1}_{p+1,\,l}$. Since there is no cohomology in antighost number $p+2$, this implies that $u^D_{p+2,\,l} = \delta\bar{u}^D_{p+3,\,l-1} + d'\tilde{u}^{D-1}_{p+2,\,l-1}$ and can be removed by trivial redefinitions: $u^D_{p+2} \to u^D_{p+2} - \delta\bar{u}^D_{p+3,\,l-1}$ and $v^{D-1}_{p+1} \to v^{D-1}_{p+1} - d\tilde{u}^{D-1}_{p+2,\,l-1}$. Repeating the steps above, one can remove all $u^D_{p+2,\,l}$ and $v^{D-1}_{p+1,\,n}$ for $l,\,n > 0$. One is left with $\lambda C^{*\,D}_{p+1} = \delta u^D_{p+2,\,0} + d'v^{D-1}_{p+1,\,0}$. The derivative argument used in the case without explicit x-dependence now leads to the desired conclusion. □

Theorem 6.4. *The cohomology groups $H^D_k(\delta|\,d)$ $(k > 1)$ vanish unless $k = D - r(D-p-1)$ for some strictly positive integer r. Furthermore, for those values of k, $H^D_k(\delta|\,d)$ has at most one non-trivial class.*

Proof: We already know that $H^D_k(\delta|\,d)$ vanishes for $k > p+1$ and that $H^D_{p+1}(\delta|\,d)$ has one non-trivial class. Let us assume that the theorem has been proved for all k's strictly greater than K (with $K < p+1$) and extend it to K. Without loss of generality we can assume that the cocycles of $H^D_K(\delta|\,d)$ take the form (up to trivial terms) $a_K = \lambda^{\mu_1\ldots\mu_{p+1-K}|\,\nu_1\ldots\nu_q} C^*_{K\;\nu_1\ldots\nu_q|\,\mu_1\ldots\mu_{p+1-K}} + \mu$, where λ does not involve the antifields and μ is at least bilinear in the antifields. Taking the Euler-Lagrange derivative of the cocycle condition with respect to C^*_{K-1} implies that $\lambda^{p+1-K}_{\nu_1\ldots\nu_q} \equiv \lambda_{\mu_1\ldots\mu_{p+1-K}|\,\nu_1\ldots\nu_q} dx^{\mu_1}\ldots dx^{\mu_{p+1-K}}$ defines an element of $H^{p+1-K}_0(d|\,\delta)$. If λ is d-trivial modulo δ, then it is straightforward to check that $\lambda C^{*\,D-p-1+K}_K$ is trivial or bilinear in the antifields. Using the isomorphism $H^{p+1-K}_0(d|\,\delta) \cong H^D_{D-p-1+K}(\delta|\,d)$, we see that λ must be trivial unless $D-p-1+K = D - r(D-p-1)$, in which case $H^D_{D-p-1+K}(\delta|\,d)$ has one non-trivial class. Since $K = D-(r+1)(D-p-1)$ is also of the required form, the theorem extends to K. □

Theorem 6.5. *Let r be a strictly positive integer. A complete set of representatives of $H^D_k(\delta|\,d)$ $(k = D - r(D-p-1) \geqslant q)$ is given by the terms of form-degree D in the expansion of all possible homogeneous polynomials $P(\tilde{H})$ of degree r in $\tilde{H}$ (or equivalently $P(\tilde{\mathcal{H}})$ of degree r in $\tilde{\mathcal{H}}$).*

The proof of this theorem is given in Appendix B.

These theorems give us a complete description of all the cohomology group $H^D_k(\delta|\,d)$ for $k \geqslant q$ (with $k > 1$).

7 Invariant cohomology of δ modulo d, $H_k^{inv}(\delta|\, d)$

In this section, we compute the set of invariant solutions a_k^D ($k \geqslant q$) of the equation $\delta a_k^D + db_{k-1}^{D-1} = 0$, up to trivial terms $a_k^D = \delta b_{k+1}^D + dc_k^{D-1}$, where b_{k+1}^D and c_k^{D-1} are invariant. This space of solutions is the invariant cohomology of δ modulo d, $H_k^{inv}(\delta|\, d)$. We first compute representatives of all the cohomology classes of $H_k^{inv}(\delta|\, d)$, then we find out the cocycles without explicit x-dependance.

Theorem 7.1. *For $k \geqslant q$, a complete set of invariant solutions of the equation $\delta a_k^D + db_{k-1}^{D-1} = 0$ is given by the polynomials in the curvature K^{q+1} and in $\tilde{\mathcal{H}}$ (modulo trivial solutions):*

$$\delta a_k^D + db_{k-1}^{D-1} = 0 \Rightarrow a_k^D = P(K^{q+1}, \tilde{\mathcal{H}})|\,_k^D + \delta\mu_{k+1}^D + d\nu_k^{D-1}\,,$$

where μ_{k+1}^D and ν_k^{D-1} are invariant forms.

Proof: From the previous section, we know that for $k \geqslant q$ the general solution of the equation $\delta a_k^D + db_{k-1}^{D-1} = 0$ is $a_k^D = Q(\tilde{\mathcal{H}})|\,_k^D + \delta m_{k+1}^D + dn_k^{D-1}$ where $Q(\tilde{\mathcal{H}})$ is a homogeneous polynomials of degree r in $\tilde{\mathcal{H}}$ (it exists only when $k = D - r(D-p-1)$). Note that m_{k+1}^D and n_k^{D-1} are not necessarily invariant. However, one can prove the following theorem (the lengthy proof of which is given in the Appendix C):

Theorem 7.2. *Let α_k^D be an invariant polynomial ($k \geqslant q$). If $\alpha_k^D = \delta m_{k+1}^D + dn_k^{D-1}$, then*

$$\alpha_k^D = R^{(s,r)}(K^{q+1}, \tilde{\mathcal{H}})|\,_k^D + \delta\mu_{k+1}^D + d\nu_k^{D-1}\,,$$

where $R^{(s,r)}(K^{q+1}, \tilde{\mathcal{H}})$ is a polynomial of degree s in K^{q+1} and r in $\tilde{\mathcal{H}}$, such that the strictly positive integers s, r satisfy $D = r(D-p-1) + k + s(q+1)$ and μ_{k+1}^D and ν_k^{D-1} are invariant forms.

As a_k^D and $Q(\tilde{\mathcal{H}})|\,_k^D$ are invariant, this theorem implies that

$$a_k^D = P^{(s,r)}(K^{q+1}, \tilde{\mathcal{H}})|\,_k^D + \delta\mu_{k+1}^D + d\nu_k^{D-1}\,,$$

where $P^{(s,r)}(K^{q+1}, \tilde{\mathcal{H}})$ is a polynomial of non-negative degree s in K^{q+1} and of strictly positive degree r in $\tilde{\mathcal{H}}$. Note that the polynomials of non-vanishing degree in K^{q+1} are trivial in $H_k^D(\delta|\, d)$ but not necessarily in $H_k^{D\,inv}(\delta|\, d)$. $\square$

Part of the solutions found in Theorem 7.1 depend explicitely on the coordinate x, because $\tilde{\mathcal{H}}|\,_0$ does. Therefore the question arises whether there exist other representatives of the same non-trivial equivalence class $[P^{(s,r)}(K^{q+1}, \tilde{\mathcal{H}})|\,_k^D] \in H_k^{D\,inv}(\delta \mid d)$ that *do not* depend explicitly on x. The answer is negative when $r > 1$. In other words, we can prove the general theorem:

Theorem 7.3. *When $r > 1$, there is no non-trivial invariant cocycle in the equivalence class $[P^{(s,r)}(K^{q+1}, \tilde{\mathcal{H}})|\,_k^D] \in H_k^{D\,inv}(\delta \mid d)$ without explicit x-dependance.*

To do so, we first prove the following lemma:

Lemma 7.1. *Let $P(K^{q+1},\tilde{\mathcal{H}})$ be a homogeneous polynomial of order s in the curvature K^{q+1} and r in $\tilde{\mathcal{H}}$. If $r \geqslant 2$, then the component $P(K^{q+1},\tilde{\mathcal{H}})|^D_k$ always contain terms of order $r-1 (\neq 0)$ in $\tilde{\mathcal{H}}|_0$.*

Proof: Indeed, $P(K^{q+1},\tilde{\mathcal{H}})$ can be freely expanded in terms of $\tilde{\mathcal{H}}|_0$ and the undifferentiated antighost forms. The Grassmann parity is the same for all terms in the expansion of $\tilde{\mathcal{H}}$, therefore the expansion is the binomial expansion up to the overall coefficient of the homogeneous polynomial and up to relative signs obtained when reordering all terms. Hence, the component $P(K^{q+1},\tilde{\mathcal{H}})|^D_k$ always contains a term that is a product of $(r-1)$ $\tilde{\mathcal{H}}|^{D-p-1}_0$'s, a single antighost $C^{*\,D-p-1+k}_k$ and s curvatures, which possesses the correct degrees as can be checked straightforwardly. □

Proof of Theorem 7.3: Let us assume that there exists a non-vanishing invariant x-independent representative $\alpha^{D\,,\,inv}_k$ of the equivalence class $[P^{(s,r)}(K^{q+1},\tilde{\mathcal{H}})|^D_k] \in H^{D\,inv}_k(\delta \mid d)$, *i.e.*

$$P^{(s,r)}(K^{q+1},\tilde{\mathcal{H}})|^D_k + \delta\rho^D_{k+1} + d\sigma^{D-1}_k = \alpha^{D\,,\,inv}_k\,, \tag{7.22}$$

where ρ^D_{k+1} and σ^{D-1}_k are invariant and allowed to depend explicitly on x.

We define the descent map $f : \alpha^n_m \rightarrow \alpha^{n-1}_{m-1}$ such that $\delta\alpha^n_m + d\alpha^{n-1}_{m-1} = 0$, for $n \leqslant D$. This map is well-defined on equivalence classes of $H^{inv}(\delta|d)$ when $m > 1$ and preserves the x-independence of a representative. Hence, going down $k-1$ steps, it is clear that the equation (7.22) implies:

$$P^{(s,r)}(K^{q+1},\tilde{\mathcal{H}})|^{D-k+1}_1 + \delta\rho^{D-k+1}_2 + d\sigma^{D-k}_1 = \alpha^{D-k+1\,,\,inv}_1\,,$$

with $\alpha^{D-k+1\,,\,inv}_1 \neq 0$.

We can decompose this equation in the polynomial degree in the fields, antifields, and all their derivatives. Since δ and d are linear operators, they preserve this degree; therefore

$$P^{(s,r)}(K^{q+1},\tilde{\mathcal{H}})|^{D-k+1}_{1,\,r+s} + \delta\rho^{D-k+1}_{2,\,r+s} + d\sigma^{D-k}_{1,\,r+s} = \alpha^{D-k+1\,,\,inv}_{1,\,r+s}\,, \tag{7.23}$$

where $r+s$ denotes the polynomial degree. The homogeneous polynomial $\alpha^{D-k+1\,,\,inv}_{1,\,r+s}$ of polynomial degree $r+s$ is linear in the antifields of antighost number equal to one, and depends on the fields only through the curvature.

Finally, we introduce the number operator N defined by

$$\begin{aligned} N &= r\,\partial_{\rho_1}\dots\partial_{\rho_r}\phi_{\mu_1\dots\mu_p\,|\,\nu_1\dots\nu_q}\,\frac{\partial}{\partial(\partial_{\rho_1}\dots\partial_{\rho_r}\phi_{\mu_1\dots\mu_p\,|\,\nu_1\dots\nu_q})} \\ &\quad + (r+1)\,\partial_{\rho_1}\dots\partial_{\rho_r}\Phi^*_A\,\frac{\partial}{\partial(\partial_{\rho_1}\dots\partial_{\rho_r}\Phi^*_A)} - x^\mu\,\frac{\partial}{\partial x^\mu} \end{aligned}$$

where $\{\Phi^*_A\}$ denotes the set of all antifields. It follows immediately that δ and d are homogeneous of degree one and the degree of $\tilde{\mathcal{H}}$ is also equal to one,

$$N(\delta) = N(d) = 1 = N(\tilde{\mathcal{H}})\,.$$

Therefore, the decomposition in N-degree of the equation (7.23) reads in N-degree equal to $n = r + 2s$,

$$P^{(s,r)}(K^{q+1}, \tilde{\mathcal{H}})|^{D-k+1}_{1,\,r+s} + \delta\rho^{D-k+1}_{2,\,r+s,\,r+2s-1} + d\sigma^{D-k}_{1,\,r+s,\,r+2s-1} = \alpha^{D-k+1\,,\,inv}_{1,\,r+s,\,r+2s} \tag{7.24}$$

and, in N–degree equal to $n > r + 2s$,

$$\delta\rho^{D-k+1}_{2,\,r+s,\,n-1} + d\sigma^{D-k}_{1,\,r+s,\,n-1} = \alpha^{D-k+1\,,\,inv}_{1,\,r+s,\,n}\,.$$

The component $\alpha^{D-k+1\,,\,inv}_{1,\,r+s,\,r+2s}$ of N-degree equal to $r+2s$ is x-independent, depends linearly on the (possibly differentiated) antighost of antifield number 1, and is of order $r+s-1$ in the (possibly differentiated) curvatures. Direct counting shows that there is no polynomial of N-degree equal to $r+2s$ satisfying these requirements when $r \geqslant 2$. Thus for $r \geqslant 2$ the component $\alpha^{D-k+1\,,\,inv}_{1,\,r+s,\,r+s}$ vanishes, and then the equation (7.24) implies that $P^{(s,r)}(K^{q+1}, \tilde{\mathcal{H}})|^{D-k+1}_{1,\,r+s}$ is trivial (and even vanishes when $s = 0$, by Theorem 6.5).

In conclusion, if $P(K^{q+1}, \tilde{\mathcal{H}})$ is a polynomial that is quadratic or more in $\tilde{\mathcal{H}}$, then there exists no non-trivial invariant representative without explicit x-dependence in the cohomology class $[P(K^{q+1}, \tilde{\mathcal{H}})]$ of $H^{inv}(\delta|d)$. $\square$

This leads us to the following theorem:

Theorem 7.4. *The invariant solutions a^D_k $(k \geqslant q)$ of the equation $\delta a^D_k + db^{D-1}_{k-1} = 0$ without explicit x-dependence are all trivial in $H^{inv}_k(\delta|\, d)$ unless $k = p + 1 - s(q + 1)$ for some non-negative integer s. For those values of k, the non-trivial representatives are given by polynomials that are linear in $C^{*\,D-p-1+k}_k$ and of order s in K^{q+1}.*

Proof: By Theorem 7.1, invariant solutions of the equation $\delta a^D_k + db^{D-1}_{k-1} = 0$ are polynomials in K^{q+1} and $\tilde{\mathcal{H}}$ modulo trivial terms. When the polynomial is quadratic or more in $\tilde{\mathcal{H}}$, then Theorem 7.3 states that there is no representative without explicit x-dependance in its cohomology class, which implies that it should be rejected. The remaining solutions are the polynomials linear in $\tilde{\mathcal{H}}|_{\,k} = C^{*\,D-p-1+k}_k$ and of arbitrary order in K^{q+1}. They are invariant and x-independent, they thus belong to the set of looked-for solutions. $\square$

8 Self-interactions

As explained in Section 3, the non-trivial first order deformations of the free theory are given by the elements of $H^{D,0}(s|\, d)$, the cohomological group of the BRST differential s in the space of local functionals in top form degree and in ghost number zero. The purpose of this section is to compute this group. As the computation is very similar to the computation of similar groups in the case of p-forms [16], gravity [17], dual gravity [18] and $[p, p]$-fields [20], we will not reproduce it here entirely and refer to the works just cited (e.g. [17]) for technical details.

We just present the main steps of the procedure and the calculations that are specific to the case of $[p, q]$-fields.

The proof is given for a single $[p, q]$-field ϕ but extends trivially to a set $\{\phi^a\}$ containing a finite number n of them (with fixed p and q) by writing some internal index $a = 1, \dots, n$ everywhere.

The group $H(s|\, d)$ is the group of solutions a of the equation $sa + db = 0$, modulo trivial solutions of the form $a = sm + dn$. The basic idea to compute such a group is to use homological perturbation techniques by expanding the quantities and the equations according to the antighost number.

Let $a^{D,0}$ be a solution of $sa^{D,0} + db^{D-1,1} = 0$ with ghost number zero and top form degree. For convenience, we will frequently omit to write the upper indices. One can expand $a(= a^{D,0})$ as $a = a_0 + a_1 + \dots + a_k$ where a_i has antighost number i. The expansion can be assumed to stop at some finite value of the antighost number under the sole hypothesis that the first-order deformation of the Lagrangian has a finite derivative order [25]. Let us recall [21] that (i) the antifield-independent piece a_0 is the deformation of the Lagrangian; (ii) the terms linear in the ghosts contain the information about the deformation of the reducibility conditions; (iii) the other terms give the information about the deformation of the gauge algebra.

Under the assumption of locality, the expansion of b also stops at some finite antighost number. Without loss of generality, one can assume that $b_j = 0$ for $j \geqslant k$. Decomposing the BRST differential as $s = \gamma + \delta$, the equation $sa + db = 0$ is equivalent to

$$\begin{aligned} \delta a_1 + \gamma a_0 + db_0 &= 0\,, \\ \delta a_2 + \gamma a_1 + db_1 &= 0\,, \\ &\vdots \\ \delta a_k + \gamma a_{k-1} + db_{k-1} &= 0\,, \\ \gamma a_k &= 0\,. \end{aligned} \tag{8.25}$$

The next step consists in the analysis of the term a_k with highest antighost number and the determination of whether it can be removed by trivial redefinitions or not. We will see in the sequel under which assumptions this can be done.

8.1 Computation of a_k for $k > 1$

The last equation of the descent (8.25) is $\gamma a_k = 0$. It implies that $a_k = \alpha_J\, \omega^J$ where α_J is an invariant polynomial and ω^J is a polynomial in the ghosts of $H(\gamma)$: $A^{(p-q,q)}_{\mu_{[q]}}$ and $D^0_{\mu_{[p+1]}}$. Inserting this expression for a_k into the second to last equation leads to the result that α_J should be an element of $H^{D,\,inv}_k(\delta|\, d)$. Furthermore, if α_J is trivial in this group, then a_k can be removed by trivial redefinitions. The vanishing of $H^{D,\,inv}_k(\delta|\, d)$ is thus a sufficient condition to remove the component a_k from a. It is however not a necessary condition, as we will see in the sequel.

We showed that non-trivial interactions can arise only if some $H_k^{D,\,inv}(\delta|\,d)$ do not vanish. The requirement that the Lagrangian should not depend explicitly on x implies that we can restrict ourselves to x-independent elements of this group. Indeed, it can be shown [28] that, when a_0 does not depend explicitly on x, the whole cocycle $a = a_0 + a_1 + \ldots + a_k$ satisfying $sa + db = 0$ is x-independent (modulo trivial redefinitions). By Theorem 7.4, $H_k^{D,\,inv}(\delta|\,d)$ contains non-trivial x-independent elements only if $k = p+1-s(q+1)$ for some non-negative integer s. The form of the non-trivial elements is then $\alpha_k^D = C_k^{*\,D-p-1+k}(K^{q+1})^s$. In order to be (possibly) non-trivial, a_k must thus be a polynomial linear in $C_k^{*\,D-p-1+k}$, of order s in the curvature K^{q+1} and of appropriate orders in the ghosts $A^{(p-q,q)}_{\mu_{[q]}}$ and $D^0_{\mu_{[p+1]}}$.

As a_k has ghost number zero, the antighost number of a_k should match its pureghost number. Consequently, as the ghosts $A^{(p-q,q)}_{\mu_{[q]}}$ and $D^0_{\mu_{[p+1]}}$ have *puregh* $= p$ and q respectively, the equation $k = np + mq$ should be satisfied for some positive integers n and m. If there is no couple of integers n, m to match k, then no a_k satisfying the equations of the descent (8.25) can be constructed and a_k thus vanishes.

In the sequel, we will consider the case where n and m satisfying $k = np + mq$ can be found and classify the different cases according to the value of n and m: (i) $n \geqslant 2$, (ii) $n = 1$, (iii) $n = 0$, $m > 1$, and (iv) $n = 0$, $m = 1$. We will show that the corresponding candidates a_k are either obstructed in the lift to a_0 or that they are trivial, except in the case (iv). In this case, a_k can be lifted but a_0 depends explicitly on x and contains more than two derivatives.

(i) Candidates with $n \geqslant 2$: The constraints $k \leqslant p + 1$ and $k = np + mq$ have no solutions[4].

(ii) Candidates with $n = 1$: The conditions $k = mq + p \leqslant p + 1$ are only satisfied for $q = 1 = m$. As shown in [18], the lift of these candidates is obstructed after one step without any additionnal assumption.

(iii) Candidates with $n = 0$, $m > 1$: For a non-trivial candidate to exist at $k = mq$, Theorem 7.4 tells us that p and q should satisfy the relation $p + 1 = mq + s(q + 1)$ for some positive or null integer s. The candidate then has the form

$$a^D_{mq} = C^{*\,D-p-1+mq}_{mq\,\nu_{[q]}}\,\omega^{\nu_{[q]}}_{(s,m)}(K, D)\,,$$

where what is meant by a polynomial $\omega_{(s,m)}$ is explained in Section 5.2.

We will show that these candidates are either trivial or that there is an obstruction to lift them up to a_0^D after q steps.

It is straightforward to check that, for $1 \leqslant j \leqslant q$, the terms

$$a^D_{mq-j} = C^{*\,D-p-1+mq-j}_{mq-j}\,\omega^{s(q+1)+j,\,mq-j}$$

[4] There is a solution in the case previously considered in [17], where $p = q = 1$, $n = 2$. As shown in [17], this solution gives rise to Einstein's theory of gravity.

satisfy the descent equations, since, as $m > 1$, all antifields $C^{*\,D-p-1+mq-j}_{mq-j}$ are invariant. The set of summed indices $\nu_{[q]}$ is implicit as well as the homogeneity degree of the generating polynomials $\omega_{(s,m)}$. We can thus lift a^D_{mq} up to $a^D_{(m-1)q}$. As $m > 1$, this is not yet a_0.

There is however no $a^D_{(m-1)q-1}$ such that

$$\gamma(a^D_{(m-1)q-1}) + \delta a^D_{(m-1)q} + d\beta^{D-1}_{(m-1)q-1} = 0\,. \tag{8.26}$$

Indeed, we have

$$\begin{aligned}\delta a^D_{(m-1)q} &= -\gamma(C^{*\,D-(s+1)(q+1)}_{(m-1)q-1}\,\omega^{(s+1)(q+1),\,(m-1)q-1}) \\ &\quad +(-)^{D-mq}\,C^{*\,D-(s+1)(q+1)}_{(m-1)q-1}\,K^{q+1}\Big[\frac{\partial^L\omega}{\partial D}\Big]^{s(q+1),(m-1)q}\,.\end{aligned}$$

Without loss of generality, we can suppose that

$$a^D_{(m-1)q-1} = C^{*\,D-(s+1)(q+1)}_{(m-1)q-1}\,\bar{a}_0^{(s+1)(q+1)} + \bar{a}^D_{(m-1)q-1}\,,$$

where there is an implicit summation over all possible coefficients $\bar{a}_0^{(s+1)(q+1)}$, and most importantly the two $\bar{a}$'s *do not*[5] depend on $C^*_{(m-1)q-1}$. Taking the Euler-Lagrange derivative of (8.26) with respect to $C^*_{(m-1)q-1}$ yields

$$\gamma(\tilde{a}_0^{(s+1)(q+1)} - \omega^{(s+1)(q+1),\,(m-1)q-1}) \propto K^{q+1}\Big[\frac{\partial^L\omega}{\partial D}\Big]^{s(q+1),(m-1)q}\,.$$

The product of non-trivial elements of $H(\gamma)$ in the r.h.s. is not γ-exact and constitutes an obstruction to the lift of the candidate, unless it vanishes. The latter happens only when the polynomial $\omega_{(s,m)}$ can be expressed as

$$\omega^{\nu_{[q]}}_{(s,m)}(K,D) = K^{q+1\,\mu_{[p+1]}}\frac{\partial^L\tilde{\omega}^{\nu_{[q]}}_{(s-1,m+1)}(K,D)}{\partial D^{\mu_{[p+1]}}}\,,$$

for some polynomial $\tilde{\omega}^{\nu_{[q]}}_{(s-1,m+1)}(K,D)$ of order $s-1$ in K^{q+1} and $m+1$ in D. However, in this case, a^D_{mq} can be removed by the trivial redefinition

$$a^D \to a^D + s(\tilde{H}_{\nu_{[q]}}\tilde{\omega}^{\nu_{[q]}}_{(s-1,m+1)}|^D)\,.$$

This completes the proof that these candidates are either trivial or that their lift is obstructed. As a consequence, they do not lead to consistent interactions and can be rejected. Let us stress that no extra assumption are needed to get this result. In the particular case $q = 1$, this had already been guessed but not been proved in [18].

[5]This is not true in the case — excluded in this paper — where $p = q = 1$ and $m = 2$: since $C^*_{(m-1)q-1} \equiv C^*_0$ has antighost number zero, the antighost number counting does not forbid that the $\bar{a}$'s depend on C^*_0. Candidates arising in this way are treated in [29] and give rise to a consistent deformation of Fierz-Pauli's theory in $D = 3$.

(iv) Candidates with $n = 0\,,\ m = 1$: These candidates exist only when the condition $p+2 = (s+1)(q+1)$ is satisfied, for some strictly positive integer s. It is useful for the analysis to write the indices explicitly:

$$a^D_q \;=\; g^{\nu_{[q]}\|\,\mu^1_{[p+1]}|\ldots|\,\mu^{s+1}_{[p+1]}}\, C^{*\,D-p-1+q}_{q\,\nu_{[q]}} \left(\prod_{i=1}^{s} K^{q+1}_{\mu^i_{[p+1]}}\right) D^0_{\mu^{s+1}_{[p+1]}}\,,$$

where g is a constant tensor.

We can split the analysis into two cases: (i) $g \to (-)^q g$ under the exchange $\mu^s_{[p+1]} \leftrightarrow \mu^{s+1}_{[p+1]}$, and (ii) $g \to (-)^{q+1} g$ under the same transformation.

In the case (i), a^D_q can be removed by adding the trivial term $s\,m^D$ where $m^D = \sum_{j=q}^{2q} m^D_j$ and

$$m^D_j = (-)^{D-q}\frac{1}{2}\, g^{\nu_{[q]}\|\,\mu^1_{[p+1]}|\ldots|\,\mu^{s+1}_{[p+1]}}\; C^{*\,D-p-1+j}_{j\;\nu[q]} \left(\prod_{i=1}^{s-1} K^{q+1}_{\mu^i_{[p+1]}}\right) \left[D_{\mu^s_{[p+1]}} D_{\mu^{s+1}_{[p+1]}}\right]^{2q+1-j}.$$

This construction does not work in the case (ii) where the symmetry of g makes m^D vanish.

In the case (ii), the candidate a^D_q can be lifted up to a^D_0:

$$a^D_0 \propto f^{\sigma_{[p+1]}\|\,\mu^1_{[p+1]}|\ldots|\,\mu^{s+1}_{[p+1]}}_{\tau_{[D-p-q-1]}} x^{\tau_1} dx^{\tau_2}\ldots dx^{\tau_{D-p-q-1}}\; K^{q+1}_{\sigma_{[p+1]}} \left(\prod_{i=1}^{s} K^{q+1}_{\mu^i_{[p+1]}}\right) D^q_{\mu^{s+1}_{[p+1]}}\,,$$

where the constant tensor f is defined by

$$f^{\sigma_{[p+1]}\|\,\mu^1_{[p+1]}|\ldots|\,\mu^{s+1}_{[p+1]}}_{\tau_{[D-p-q-1]}} \equiv g^{\nu_{[q]}\|\,\mu^1_{[p+1]}|\ldots|\,\mu^{s+1}_{[p+1]}}\; \epsilon^{\sigma_{[p+1]}}{}_{\nu_{[q]}\tau_{[D-p-q-1]}}\,.$$

Let us first note that this deformation does not affect the gauge algebra, since it is linear in the ghosts.

The Lagrangian deformation a^D_0 depends explicitly on x, which is not a contradiction with translation-invariance of the physical theory if the x-dependance of the Lagrangian can be removed by adding a total derivative and/or a δ-exact term. If it were the case, a^D_0 would have the form $a^D_0 = xG(\ldots) + x^\alpha d(\ldots)_\alpha$. We have no complete proof that a^D_0 does not have this form, but it is not obvious and we think it very unlikely. In any case, this deformation is ruled out by the requirement that the deformation of the Lagrangian contains at most two derivatives.

To summarize the results obtained in this section, we have proved that, under the hypothesis of translation-invariance of the first-order vertex a^D_0, all a^D_k $(k>1)$ can be removed by trivial redefinitions of a, except when $p+2 = (s+1)(q+1)$ for some positive integer s. In that case, the supplementary assumption that the deformed Lagrangian contains no more than two derivatives is needed to reach the same conclusion, and the only possible deformation (without the latter assumption) does not modify the gauge algebra.

8.2 Computation of a_1

The term a_1 vanishes without any further assumption when $q > 1$. Indeed, when $q > 1$, the vanishing of the cohomology of γ in *puregh* 1 implies that there is no non-trivial a_1 .

This is not true when $q = 1$, as there are some non-trivial cocycles with pureghost number equal to one. However, it can be shown [18] that any non-trivial a_1^D leads to a deformation of the Lagrangian with at least four derivatives.

8.3 Computation of a_0

This leaves us with the problem of solving the equation $\gamma a_0^D + d\, b_0^{D-1} = 0$ for a_0^D . Such solutions correspond to deformations of the Lagrangian that are invariant up to a total derivative. Proceeding as in [20] and asking for Lorentz invariance and that a_0^D should not contain more than two derivatives leaves only[6] the Lagrangian itself. This deformation is of course trivial.

9 Conclusions

Assembling the results of the present paper ($p \neq q$) with those previously obtained in [20] ($p = q \neq 1$), we can state general conclusions for $[p, q]$-tensor gauge fields where p and q are now arbitrary but both different from one. Under the hypothesis of locality and translation invariance, there is no smooth deformation of the free theory that modifies the gauge algebra, which remains Abelian. This result strengthens the conclusions of [18], as no condition on the number of derivative is needed any longer. Furthermore, for $q > 1$, when there is no positive integer s such that $p + 2 = (s + 1)(q + 1)$, there exists also no smooth deformation that alters the gauge transformations. Finally, if one excludes deformations that involve more than two derivatives in the Lagrangian and are not Lorentz-invariant, then the only smooth deformation of the free theory is a cosmological-like term for $p = q$ [20].

These no-go results complete the search for self-interactions of $[p, q]$-tensor gauge fields. It is still an open question whether interactions are possible between N different $[p, q]$-type fields (where "different" means $[p_1, q_1] \neq [p_2, q_2]$ for $N = 2$), or with other types of fields.

As a conclusion, one can reformulate the results in more physical terms by saying that no analogue of Yang-Mills nor Einstein theories seems to exist for more exotic fields (at least not in the range of local perturbative theories).

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[6]When $p = q$, there exists also a cosmological-like term [20]: $a_0 = \Lambda \eta_{\mu_1\nu_1} \dots \eta_{\mu_p\nu_p} \phi^{\mu_1\dots\mu_p|\nu_1\dots\nu_p}$.

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Appendices

A Going to the Light-cone

Theorem A.1. *Let K be a tensor in the irreducible representation $[p{+}1,q{+}1]$ of $O(D{-}1,1)$. The space of such harmonic multiforms K, i.e. solutions of*

$$\left.\begin{array}{ll}\partial_{[\mu_0}K_{\mu_1\dots\mu_{p+1}]|\nu_1\dots\nu_{q+1}}=0=K_{\mu_1\dots\mu_{p+1}|[\nu_1\dots\nu_{q+1},\nu_0]} & (\textit{closed})\\ \partial^{\mu_1}K_{\mu_1\dots\mu_{p+1}|\nu_1\dots\nu_{q+1}}=0=\partial^{\nu_1}K_{\mu_1\dots\mu_{p+1}|\nu_1\dots\nu_{q+1}} & (\textit{coclosed})\end{array}\right\}\Longrightarrow\quad \Box K=0$$

is a unitary irreducible module of $O(D-2)$ associated to the Young diagram $[p,q]$.

Proof : Since $\Box K(x)=0$ then, after Fourier transform, $K(p)\neq 0$ iff $p^2=0$. In the light-cone frame, the light-like momentum p^μ decomposes into

$$p^\mu=(p^+,p^-,\underbrace{0,\dots,0}_{D-2}),\quad p^-=0\,.$$

(i) The condition that K is closed implies

$$\left\{\begin{array}{l}p_\mu\varepsilon^{\mu\nu_1\dots\nu_{D-p-2}\mu_1\dots\mu_{p+1}}K_{\mu_1\dots\mu_{p+1}|\alpha_1\dots\alpha_{q+1}}=0\,,\\ p_\mu\varepsilon^{\mu\nu_1\dots\nu_{D-q-2}\mu_1\dots\mu_{q+1}}K_{\alpha_1\dots\alpha_{p+1}|\mu_1\dots\mu_{q+1}}=0\,,\end{array}\right.$$

i.e.

$$\left\{\begin{array}{l}\varepsilon^{-\nu_1\dots\nu_{D-p-2}\mu_1\dots\mu_{p+1}}K_{\mu_1\dots\mu_{p+1}|\alpha_1\dots\alpha_{q+1}}=0\,,\\ \varepsilon^{-\nu_1\dots\nu_{D-q-2}\mu_1\dots\mu_{q+1}}K_{\alpha_1\dots\alpha_{p+1}|\mu_1\dots\mu_{q+1}}=0\,.\end{array}\right.$$

The latin indices will run over the $D-2$ transverse values. Assigning $\nu_1=+$, $\nu_2=j_2$, $\cdots$, $\nu_{D-\ell-2}=j_{D-\ell-2}$ (where $\ell=p$ or q respectively), one finds

$$K_{i_1\dots i_{p+1}|\alpha_1\dots\alpha_{q+1}}=0=K_{\alpha_1\dots\alpha_{p+1}|i_1\dots i_{q+1}}\,.$$

In other words, K vanishes whenever one of its columns contains only transverse indices.

(ii) The fact that K is coclosed on-shell implies

$$p^+K_{+\mu_2\dots\mu_{p+1}|\alpha_1\dots\alpha_{q+1}}=0=p^+K_{\alpha_1\dots\alpha_{p+1}|+\mu_2\dots\mu_{q+1}}\,,$$

i.e.

$$K_{+\mu_2\dots\mu_{p+1}|\alpha_1\dots\alpha_{q+1}}=0=K_{\alpha_1\dots\alpha_{p+1}|+\mu_2\dots\mu_{q+1}}\,.$$

In other words, K vanishes whenever one of its columns contains a "+" index.

Once it has been observed that each column of K must contain at least one "$-$" index and no "$+$" index, one finds that the tensor

$$\phi_{i_1\dots i_p|j_1\dots j_q} \equiv \frac{(p+1)(q+1)}{p_-^2} K_{-i_1\dots i_p|j_1\dots j_q-}$$

obeys

$$\begin{aligned}
0 &= \frac{p+2}{p_-^2} K_{[-i_1\dots i_p|j_1]\dots j_q-} = \phi_{[i_1\dots i_p|j_1]\dots j_q}\,, \\
0 &= \eta^{\mu_1\nu_1} K_{\mu_1\mu_2\dots\mu_{p+1}|\nu_1\dots\nu_{q+1}} \\
&\Rightarrow\ 0 = \frac{(p+1)(q+1)}{p_-^2}\,\delta^{i_1j_1} K_{-i_1i_2\dots i_p|j_1\dots j_q-} = \delta^{i_1j_1}\phi_{i_1i_2\cdots i_p|j_1\cdots j_q}\,.
\end{aligned}$$

□

B Proof of Theorem 6.5

In this appendix, we give the proof of Theorem 6.5:

Let r be a strictly positive integer. A complete set of representatives of $H^D_k(\delta|\,d)$ ($k>1$ and $k = D - r(D-p-1) \geqslant q$) is given by the terms of form-degree D in all homogeneous polynomials $P^{(r)}(\tilde{H})$ of degree r in $\tilde{H}$ (or equivalently $P(\tilde{\mathcal{H}})$ of degree r in $\tilde{\mathcal{H}}$).

It is obvious from the definition of $\tilde{H}$ and from equation (3.10) that the term of form-degree D in $P^{(r)}(\tilde{H})$ has the right antighost number and is a cocycle of $H^D_k(\delta|\,d)$. Furthermore, as $\tilde{\mathcal{H}} = \tilde{H} + d(\dots)$, $P^{(r)}(\tilde{\mathcal{H}})$ belongs to the same cohomology class as $P^{(r)}(\tilde{H})$ and can as well be chosen as a representative of this class. To prove the theorem, it is then enough, by Theorem 6.4, to prove that the cocycle $P^{(r)}(\tilde{H})|^{\,D}_{\,k}$ is non-trivial. The proof is by induction: we know the theorem to be true for $r=1$ by Theorem 6.3, supposing that the theorem is true for $r-1$, (*i.e.* $[P^{(r-1)}(\tilde{H})]^D_{k+D-p-1}$ is not trivial in $H^D_{k+D-p-1}(\delta|d)$) we prove that $[P^{(r)}(\tilde{H})]^D_k$ is not trivial either.

Let us assume that $[P^{(r)}(\tilde{H})]^D_k$ is trivial: $[P^{(r)}(\tilde{H})]^D_k = \delta(u_{k+1}d^Dx) + dv^{D-1}_k$. We take the Euler-Lagrange derivative of this equation with respect to $C^*_{k,\mu_{[q]}|\nu_{[p+1-k]}}$. For $k>q$, it reads:

$$\alpha_{\mu_{[q]}|\,\nu_{[p+1-k]}} = (-)^k\delta(Z_{1\ \mu_{[q]}|\,\nu_{[p+1-k]}}) - Z_{0\ \mu_{[q]}|\,[\nu_{[p-k]},\nu_{p+1-k}]}\,, \tag{B.27}$$

where

$$\begin{aligned}
\alpha_{\mu_{[q]}|\,\nu_{[p+1-k]}}d^Dx &\equiv \frac{\delta^L[P^{(r)}(\tilde{H})]^D_k}{\delta C^{*\ \mu_{[q]}|\,\nu_{[p+1-k]}}_k}\,, \\
Z_{k+1-j\ \mu_{[q]}|\,\nu_{[p+1-j]}} &\equiv \frac{\delta^L u_{k+1}}{\delta C^{*\ \mu_{[q]}|\,\nu_{[p+1-j]}}_j}\,,\ \text{ for } j = k, k+1\,.
\end{aligned}$$

For $k = q$, there is an additional term:

$$\alpha_{\mu_{[q]}|\,\nu_{[p+1-q]}} = (-)^q \delta(Z_{1\ \mu_{[q]}|\,\nu_{[p+1-q]}}) - (Z_{0\ \mu_{[q]}|\,[\nu_{[p-q]},\nu_{p+1-q}]} - Z_{0\ [\mu_{[q]}|\,\nu_{[p-q]},\nu_{p+1-q}]})\,. \quad \text{(B.28)}$$

The origin of the additional term lies in the fact that $C_q^{*\ \mu_{[q]}|\,\nu_{[p+1-q]}}$ does not possess all the irreducible components of $[q]\otimes[p+1-q]:$ the completely antisymmetric component $[p+1]$ is missing. Taking the Euler-Lagrange derivative with respect to this field thus involves projecting out this component.

We will first solve the equation (B.27) for $k > q$, then come back to (B.28) for $k = q$.

Explicit computation of $\alpha_{\mu_{[q]}|\,\nu_{[p+1-k]}}$ for $k > q$ yields:

$$\alpha_{\mu_{[q]}|\,\nu_{[p+1-k]}} = [\tilde{H}^{\rho^1_{[q]}}]_{0,\,\sigma^1_{[D-p-1]}} \dots [\tilde{H}^{\rho^{r-1}_{[q]}}]_{0,\,\sigma^{r-1}_{[D-p-1]}} a_{\mu_{[q]}|\rho^1_{[q]}|\dots|\rho^{r-1}_{[q]}} \delta^{[\sigma^1_{[D-p-1]}\dots\sigma^{r-1}_{[D-p-1]}]}_{\nu_{[p+1-k]}}\,,$$

where a is a constant tensor and the notation $[A]_{k,\,\nu_{[p]}}$ means the coefficient $A_{k,\,\nu_{[p]}}$, with antighost number k, of the p-form component of $A = \sum_{k,l} A_{k,\,\nu_{[l]}} dx^{\nu_1}\dots dx^{\nu_l}$. Considering the indices $\nu_{[p+1-k]}$ as form indices, (B.27) reads:

$$\begin{aligned}\alpha^{p+1-k}_{\mu_{[q]}} &= [\tilde{H}^{\rho^1_{[q]}}]^{D-p-1}_0 \dots [\tilde{H}^{\rho^{r-1}_{[q]}}]^{D-p-1}_0 a_{\mu_{[q]}|\rho^1_{[q]}|\dots|\rho^{r-1}_{[q]}} = \Big[\prod_{i=1}^{(r-1)} \tilde{H}^{\rho^i_{[q]}}\Big]^{p+1-k}_0 a_{\mu_{[q]}|\rho^1_{[q]}|\dots|\rho^{r-1}_{[q]}} \\ &= (-)^k \delta(Z^{p+1-k}_{1\ \mu_{[q]}}) + (-)^{p-k+1} d\, Z^{p-k}_{0\ \mu_{[q]}}\,.\end{aligned}$$

The latter equation is equivalent to

$$\Big[\prod_{i=1}^{(r-1)} \tilde{H}^{\rho^i_{[q]}}\Big]^{D}_{D-p-1+k} a_{\mu_{[q]}|\rho^1_{[q]}|\dots|\rho^{r-1}_{[q]}} = \delta(\dots) + d(\dots)\,,$$

which contradicts the induction hypothesis. The assumption that $[P^{(r)}(\tilde{H})]^D_k$ is trivial is thus wrong, which proves the theorem for $k > q$.

The philosophy of the resolution of (B.28) for $k = q$ is inspired by the proof of Theorem 3.3 in [20] and goes as follows: first, one has to constrain the last term of (B.28) in order to get an equation similar to the equation (B.27) treated previously, then one solves this equation in the same way as for $k > q$.

Let us constrain the last term of (B.28). Equation (B.28) and explicit computation of $\alpha_{\mu_{[q]}|\,\nu_{[p+1-k]}}$ imply

$$\begin{aligned}\partial_{[\nu_{p+1-q}}\alpha_{\mu_{[q]}|\,\nu_{[p-q]}]\lambda} &= (-)^q\delta(\partial_{[\nu_{p+1-q}}Z_{1\ \mu_{[q]}|\,\nu_{[p-q]}]\lambda}) - b\,\partial_{[\nu_{p+1-q}}Z_{0\ \mu_{[q]}|\,\nu_{[p-q]}],\lambda} \\ &\approx b\,\partial_\lambda([\tilde{H}^{\rho^1_{[q]}}]_{0,\,\sigma^1_{[D-p-1]}} \dots [\tilde{H}^{\rho^{r-1}_{[q]}}]_{0,\,\sigma^{r-1}_{[D-p-1]}} \delta^{[\sigma^1_{[D-p-1]}\dots\sigma^{r-1}_{[D-p-1]}]}_{[\nu_{[p+1-k]}} a_{\mu_{[q]}]|\rho^1_{[q]}|\dots|\rho^{r-1}_{[q]}})\end{aligned}$$

where $b = \frac{q}{(p+1)(p+1-q)}$. By the isomorphism $H^0_0(d|\delta)/\mathbb{R} \cong H^D_D(\delta|d) \cong 0$, the latter equation implies

$$Z_{0\ [\mu_{[q]}|\,\nu_{[p-q]},\nu_{p+1-q}]} \approx -[\tilde{H}^{\rho^1_{[q]}}]_{0,\,\sigma^1_{[D-p-1]}} \dots [\tilde{H}^{\rho^{r-1}_{[q]}}]_{0,\,\sigma^{r-1}_{[D-p-1]}} a_{\mu_{[q]}|\rho^1_{[q]}|\dots|\rho^{r-1}_{[q]}} \delta^{[\sigma^1_{[D-p-1]}\dots\sigma^{r-1}_{[D-p-1]}]}_{\nu_{[p+1-k]}}$$

(the constant solutions are removed by considering the equation in polynomial degree $r-1$ in the fields and antifields.). Inserting this expression for $Z_{0\,[\mu_{[q]}|\,\nu_{[p-q]},\nu_{p+1-q}]}$ into (B.28) and redefining Z_1 in a suitable way yields (B.27) for $k=q$. The remaining of the proof is then the same as for $k>q$. $\square$

C Proof of Theorem 7.2

In this appendix, we give the complete (and lengthy) proof of Theorem 7.2:

Let a_k^D be an invariant polynomial. If $a_k^D = \delta b_{k+1}^D + dc_k^{D-1}$, then

$$a_k^D = P_{(s,r)}(K^{q+1},\tilde{\mathcal{H}})|\,{}_k^D + \delta\mu_{k+1}^D + d\nu_k^{D-1}\,,$$

where $P_{(s,r)}(K^{q+1},\tilde{\mathcal{H}})$ is a polynomial of degree s in K^{q+1} and r in $\tilde{\mathcal{H}}$, such that the integers $s,r\geqslant 1$ satisfy $D=r(D-p-1)+k+s(q+1)$ and μ_{k+1}^D and ν_k^{D-1} are invariant polynomials.

The proof is by induction and follows closely the steps of the proof of similar theorems in the case of 1-forms [24, 25], p-forms [26], gravity [17] or $[p,p]$-fields [20].

There is a general procedure to prove that the theorem 7.2 holds for $k>D$, that can be found e.g. in [17] and will not be repeated here. We assume that the theorem has been proved for any $k^{'}>k$, and show that it is still valid for k .

The proof of the induction step is rather lengthy and is decomposed into several steps:

- the Euler-Lagrange derivatives of a_k with respect to the fields ϕ and C_j^* $(1\leqslant j\leqslant p+1)$ are computed in terms of the Euler-Lagrange derivatives of b_{k+1} (section C.1);
- it is shown that the Euler-Lagrange derivatives of b_{k+1} can be replaced by invariant quantities in the expression for the Euler-Lagrange derivative of a_k with the lowest antighost number, up to some additionnal terms (section C.2);
- the previous step is extended to all the Euler-Lagrange derivatives of a_k (section C.3);
- the Euler-Lagrange derivative of a_k with respect to the field ϕ is reexpressed in terms of invariant quantities (section C.4);
- an homotopy formula is used to reconstruct a_k from its Euler-Lagrange derivatives (section C.5).

C.1 Euler-Lagrange derivatives of a_k

We define

$$\begin{aligned} Z_{k+1-j\;\mu_{[q]}|\,\nu_{[p+1-j]}} &= \frac{\delta^L b_{k+1}}{\delta C_j^{*\;\mu_{[q]}|\,\nu_{[p+1-j]}}}\,, \qquad 1\leqslant j\leqslant p+1\,,\\ Y_{k+1}^{\mu_{[p]}|\,\nu_{[q]}} &= \frac{\delta^L b_{k+1}}{\delta\phi_{\mu_{[p]}|\,\nu_{[q]}}}\,. \end{aligned}$$

Then, the Euler-Lagrange derivatives of a_k are given by

$$
\begin{aligned}
\frac{\delta^L a_k}{\delta C^{*\,\mu_{[q]}}_{p+1}} &= (-)^{p+1}\delta Z_{k-p\;\mu_{[q]}}\,, && \text{(C.29)}\\
\frac{\delta^L a_k}{\delta C^{*\,\mu_{[q]}|\,\nu_{[p+1-j]}}_j} &= (-)^j\delta Z_{k+1-j\;\mu_{[q]}|\,\nu_{[p+1-j]}} - Z_{k-j\;\mu_{[q]}|\,[\nu_{[p-j]},\nu_{p+1-j}]}\;,\; q<j\leqslant p\,, &&\\
\frac{\delta^L a_k}{\delta C^{*\,\mu_{[q]}|\,\nu_{[p+1-j]}}_j} &= (-)^j\delta Z_{k+1-j\;\mu_{[q]}|\,\nu_{[p+1-j]}} - Z_{k-j\;\mu_{[q]}|\,[\nu_{[p-j]},\nu_{p+1-j}]}\big|_{\,sym\,of\,C^*_j}\;, 1\leqslant j\leqslant q\,, &&\\
\frac{\delta^L a_k}{\delta\phi^{\mu_{[p]}|\,\nu_{[q]}}} &= \delta Y_{k+1\;\mu_{[p]}|\,\nu_{[q]}} + \beta D_{\mu_{[p]}|\,\nu_{[q]}|\,\rho_{[p]}|\,\sigma_{[q]}} Z_k^{\sigma_{[q]}|\,\rho_{[p]}}\,, && \text{(C.30)}
\end{aligned}
$$

where $\beta\equiv(-)^{(q+1)(p+\frac{q}{2})}\frac{(p+1)!}{q!(p-q+1)!}$, and $D^{\mu_{[p]}|}{}_{\nu_{[q]}|\,\rho_{[p]}|}{}^{\sigma_{[q]}}\equiv\frac{1}{(p+1)!q!}\,\delta^{[\sigma_{[q]}\alpha\mu_{[p]}]}_{[\nu_{[q]}\beta\rho_{[p]}]}\partial_\alpha\partial^\beta$ is the second-order self-adjoint differential operator defined by $G_{\mu_{[p]}|\,\nu_{[q]}}\equiv D_{\mu_{[p]}|\,\nu_{[q]}|\,\rho_{[p]}|\,\sigma_{[q]}}\;C^{\rho_{[p]}|\,\sigma_{[q]}}$.

As in Appendix B, the projection on the symmetry of the indices of C^*_j is needed when $j\leqslant q$, since in that case the variables C^*_j do not possess all the irreducible components of $[q]\otimes[p+1-j]$, but only those where the length of the first column is smaller or equal to p. When $j>q$, the projection is trivial.

C.2 Replacing Z by an invariant in the Euler-Lagrange derivative of a_k with the lowest antighost number

We should first note that, when $k<p+1$, some of the Euler-Lagrange derivatives of a_k vanish identically: indeed, as there is no negative antighost-number field, a_k cannot depend on C^*_j if $j>k$. Some terms on the r.h.s. of (C.29)-(C.30) also vanish: Z_{k+1-j} vanishes when $j>k+1$. This implies that the $p+1-k$ top equations of (C.29)-(C.30) are trivially satisfied: the $p-k$ first equations involve only vanishing terms, and the $(p-k+1)$th involves in addition the δ of an antighost-zero term, which also vanishes trivially. The first non-trivial equation is then

$$
\frac{\delta^L a_k}{\delta C^*_{k\;\mu_{[q]}|\,\nu_{[p+1-k]}}} = (-)^k\delta(Z_{1\;\mu_{[q]}|\,\nu_{[p+1-k]}}) - Z_{0\;\mu_{[q]}|\,[\nu_{[p-k]},\nu_{p+1-k}]}\big|_{\,sym\,of\,C^*_k}\,. \qquad \text{(C.31)}
$$

Let us now define $[T^q_{\rho_{[p+1]}}]_{\nu_{[q]}}\equiv(-)^q\partial_{[\rho_1}\phi_{\rho_2\ldots\rho_{p+1}]|\nu_{[q]}}$. We will prove the following lemma for $k\geqslant q$:

Lemma C.1. *In the first non-trivial equation of the system (C.29)-(C.30) (i.e. (C.29) when $k\geqslant p+1$ and (C.31) when $p+1>k\geqslant q$), respectively Z_{k-p} or Z_1 satisfies*

$$
\begin{aligned}
Z_{l\;\mu_{[q]}|\,\nu_{[p+l-k]}} &= Z'_{l\;\mu_{[q]}|\,\nu_{[p+l-k]}} + (-)^{k-l}\delta\beta_{l+1\;\mu_{[q]}|\,\nu_{[p+l-k]}} + \beta_{l\;\mu_{[q]}|\,[\nu_{[p+l-k-1]},\nu_{p+l-k}]}\big|_{\,sym\,of\,C^*_{k-l+1}}\\
&+\; A_l\Big[P^{(n)}_{\mu_{[q]}}(\tilde{\mathcal{H}}) + \frac{1}{s}T^q_{\rho_{[p+1]}}\frac{\partial^L R^{(s,r)}_{\mu_{[q]}}(K^{q+1},\tilde{\mathcal{H}})}{\partial K^{q+1}_{\rho_{[p+1]}}}\Big]_{l,\,\nu_{[p+l-k]}}\big|_{\,sym\,of\,C^*_{k-l+1}}, \qquad \text{(C.32)}
\end{aligned}
$$

where Z'_l is invariant, the β_l's are at least linear in $\mathcal{N}$ and possess the same symmetry of indices as Z_{l-1} , $A_l \equiv (-)^{lp+p+1+\frac{l(l+1)}{2}}$, $P^{(n)}$ is a polynomial of degree n in $\tilde{\mathcal{H}}$ and $R^{(s,r)}$ is a polynomial of degree s in K^{q+1} and r in $\tilde{\mathcal{H}}$. The polynomials are present only when $p-k=n(D-p-1)$ or $p+1-k=s(q+1)+r(D-p-1)$ respectively.

Moreover, when $p+1>k\geqslant q$, the first non-trivial equation can be written

$$\begin{aligned}\frac{\delta^L a_k}{\delta C^*_{k\,\mu_{[q]}|\,\nu_{[p+1-k]}}} &= (-)^k\delta Z'_{1\,\mu_{[q]}|\,\nu_{[p+1-k]}} - Z'_{0\,\mu_{[q]}|\,[\nu_{[p-k]},\nu_{p+1-k}]}|_{\,sym\,of\,C^*_k}\\ &+\Big([Q^{(m)}_{\mu_{[q]}}(K^{q+1})]_{\nu_{[p+1-k]}}+(-)^k[R^{(s,r)}_{\mu_{[q]}}(K^{q+1},\tilde{\mathcal{H}})]_{0,\,\nu_{[p+1-k]}}\Big)|_{\,sym\,of\,C^*_k}\,,\end{aligned}$$

where Z'_0 is an invariant and $Q^{(m)}_{\mu_{[q]}}(K^{q+1})$ is a polynomial of degree m in K^{q+1}, present only when $p+1-k=m(q+1)$.

The lemma will be proved in the sections C.2.1–C.2.3 respectively for the cases $k\geqslant p+1$, $q<k<p+1$ and $k=q$.

C.2.1 Proof of Lemma C.1 for $k\geqslant p+1$

As $k-p>0$, there is no trivially satisfied equation and we start with the top equation of (C.29)–(C.30).

The lemma C.1 is a direct consequence of the well-known Lemma C.2 (see e.g. [17]):

Lemma C.2. *Let α be an invariant local form that is δ-exact, i.e. $\alpha=\delta\beta$. Then $\beta=\beta'+\delta\sigma$, where β' is invariant and we can assume without loss of generality that σ is at least linear in the variables of $\mathcal{N}$.*

C.2.2 Proof of Lemma C.1 for $q<k<p+1$

The first non-trivial equation is (as $k>q$):

$$\frac{\delta^L a_k}{\delta C^*_{k\,\mu_{[q]}|\,\nu_{[p+1-k]}}} = (-)^k\delta(Z_{1\,\mu_{[q]}|\,\nu_{[p+1-k]}}) - Z_{0\,\mu_{[q]}|\,[\nu_{[p-k]},\nu_{p+1-k}]}\,. \tag{C.33}$$

We will first prove that Z_1 has the required form, then we will prove the the first non-trivial equation can indeed be reexpressed as stated in Lemma C.1.

First part: Defining $\alpha_{0\,\mu_{[q]}|\nu_{[p+1-k]}}\equiv\frac{\delta^L a_q}{\delta C^*_{q\,\mu_{[q]}|\,\nu_{[p+1-q]}}}$, the above equation can be written as

$$\alpha_0^{p+1-k}=(-)^k\delta(Z_1^{p+1-k})+(-)^{p+1-k}dZ_0^{p-k}\,, \tag{C.34}$$

where we consider the indices $\nu_{[p+1-k]}$ as form-indices and omit to write the indices $\mu_{[q]}$. Acting with d on this equation yields $d\alpha_0^{p+1-k}=(-)^{k+1}\delta(dZ_1^{p+1-k})$. Due to Lemma C.2, this implies that

$$\alpha_1^{p+2-k}=dZ_1^{p+1-k}+\delta Z_2^{p+2-k}\,, \tag{C.35}$$

for some invariant α_1^{p+2-k} and some Z_2^{p+2-k}. These steps can be reproduced to build a descent of equations ending with

$$\alpha^D_{D-p-1+k} = dZ^{D-1}_{D-p-1+k} + \delta Z^D_{D-p+k}\,,$$

where $\alpha^D_{D-p-1+k}$ is invariant. As $D-p-1+k > k$, the induction hypothesis can be used and implies

$$\alpha^D_{D-p-1+k} = dZ'^{\,D-1}_{D-p-1+k} + \delta Z'^{\,D}_{D-p+k} + [R(K^{q+1},\tilde{\mathcal{H}})]^D_{D-p-1+k}\,,$$

where $Z'^{\,D}_{D-p+k}$ and $Z'^{\,D-1}_{D-p-1+k}$ are invariant, and $R(K^{q+1},\tilde{\mathcal{H}})$ is a polynomial of order s in K^{q+1} and r in $\tilde{\mathcal{H}}$ (with $r,s>0$), present when $p+1-k = s(q+1)+r(D-p-1)$. This equation can be lifted and implies that

$$\alpha_1^{p+2-k} = dZ'^{\,p+1-k}_1 + \delta Z'^{\,p+2-k}_2 + [R(K^{q+1},\tilde{\mathcal{H}})]^{p+2-k}_1\,,$$

for some invariant quantities $Z'^{\,p+1-k}_1$ and $Z'^{\,p+2-k}_2$. Substracting the last equation from (C.35) yields

$$d\Big(Z_1^{p+1-k} - Z'^{\,p+1-k}_1 - \frac{1}{s}T^q\Big[\frac{\partial^L R(K^{q+1},\tilde{\mathcal{H}})}{\partial K^{q+1}}\Big]_1^{p+1-k-q}\Big) + \delta(\ldots) = 0\,.$$

As $H_1^{p+1-k}(d|\,\delta) \cong H^D_{D-(p-k)}(\delta|\,d)$, by Theorem 6.5 the solution of this equation is

$$Z_1^{p+1-k} = Z'^{\,p+1-k}_1 + \frac{1}{s}T^q\Big[\frac{\partial^L R(K^{q+1},\tilde{\mathcal{H}})}{\partial K^{q+1}}\Big]_1^{p+1-k-q} + d\beta_1^{p-k} + \delta\beta_2^{p+1-k} + [P^{(n)}(\tilde{\mathcal{H}})]_1^{p+1-k}\,,$$

where the last term is present only when $p-k = n(D-p-1)$.

This proves the first part of the induction basis, regarding Z_1.

Second part: We insert the above result for Z_1 into (C.34). Knowing that $\delta([P(\tilde{\mathcal{H}})]_1^{p+1-k}) + d([P(\tilde{\mathcal{H}})]_0^{p-k}) = 0$ and defining

$$W_0^{p-k} = (-)^{k+1}\Big((-)^p Z_0^{p-k} + \delta\beta_1^{p-k} + [P^{(n)}(\tilde{\mathcal{H}})]_0^{p-k} + \frac{1}{s}T^q\Big[\frac{\partial^L R(K^{q+1},\tilde{\mathcal{H}})}{\partial K^{q+1}}\Big]_0^{p-k-q}\Big)\,,$$

we get

$$\alpha_0^{p+1-k} = (-)^k\delta(Z'^{\,p+1-k}_1) + d(W_0^{p-k}) + (-)^k[R(K^{q+1},\tilde{\mathcal{H}})]_0^{p-k}\,.$$

Thus $d(W_0^{p-k})$ is an invariant and the invariant Poincaré Lemma 5.1 then states that

$$d(W_0^{p-k}) = d(Z'^{\,p-k}_0) + Q(K^{q+1})$$

for some invariant $Z'^{\,p-k}_0$ and some polynomial in K^{q+1}, $Q(K^{q+1})$. This straightforwardly implies

$$\alpha_0^{p+1-k} = (-)^k\delta(Z'^{\,p+1-k}_1) + d(Z'^{\,p-k}_0) + Q(K^{q+1}) + (-)^k[R(K^{q+1},\tilde{\mathcal{H}})]_0^{p-k}\,,$$

which completes the proof of Lemma C.1 for $q<k<p+1$. $\square$

C.2.3 Proof of Lemma C.1 for $k = q$

The first non-trivial equation is

$$\frac{\delta^L a_q}{\delta C^*_{q\,\mu_{[q]}|\,\nu_{[p+1-q]}}} = (-)^q \delta(Z_{1\,\mu_{[q]}|\,\nu_{[p+1-q]}}) - (Z_{0\,\mu_{[q]}|\,[\nu_{[p-q]},\nu_{p+1-q}]} - Z_{0\,[\mu_{[q]}|\,\nu_{[p-q]},\nu_{p+1-q}]})\,. \quad (C.36)$$

This equation is different from the equations treated in the previous cases because the operator acting on Z_0 cannot be seen as a total derivative, since it involves the projection on a specific Young diagram. The latter problem was already faced in the $[p,p]$-case and the philosophy of the resolution goes as follows [20]:

(1) one first constrains the last term of (C.36) to get an equation similar to Equation (C.31) treated previously,

(2) one solves it in the same way as for $q < k < p+1$.

We need the useful lemma C.3, proved in [20].

Lemma C.3. *If α^1_0 is an invariant polynomial of antighost number 0 and form degree 1 that satisfies $\alpha^1_0 = \delta Z^1_1 + dW^0_0$, then, for some invariant polynomials Z'^1_1 and W'^0_0, $Z^1_1 = Z'^1_1 + \delta\phi^1_2 + d\chi^0_1$ and $W^0_0 = W'^0_0 + \delta\chi^0_1$.*

As explained above, we now constrain the last term of (C.36). Equation (C.36) implies

$$\partial_{[\rho}\alpha_{0\,\mu_{[q]}|\,\nu_{[p-q]}]\nu_{p+1-q}} = (-)^q\delta(\partial_{[\rho}Z_{1\,\mu_{[q]}|\,\nu_{[p-q]}]\nu_{p+1-q}}) - b\,\partial_{[\rho}Z_{0\,\mu_{[q]}|\,\nu_{[p-q]}],\nu_{p+1-q}}\,,$$

where $b \equiv \frac{q}{(p+1)(p+1-q)}$. Defining

$$\begin{aligned}
\tilde{\alpha}^1_{0\,[\rho\mu_{[q]}\nu_{[p-q]}]} &= \partial_{[\rho}\alpha_{0\,\mu_{[q]}|\,\nu_{[p-q]}]\nu_{p+1-q}}dx^{\nu_{p+1-q}}\,,\\
\tilde{Z}^1_{1\,[\rho\mu_{[q]}\nu_{[p-q]}]} &= (-)^q\partial_{[\rho}Z_{1\,\mu_{[q]}|\,\nu_{[p-q]}]\nu_{p+1-q}}dx^{\nu_{p+1-q}}\,,\\
\tilde{W}^0_{0\,[\rho\mu_{[q]}\nu_{[p-q]}]} &= -a\,\partial_{[\rho}Z_{0\,\mu_{[q]}|\,\nu_{[p-q]}]}\,,
\end{aligned}$$

and omitting to write the indices $[\rho\mu_{[q]}\nu_{[p-q]}]$, the above equation reads $\tilde{\alpha}^1_0 = \delta\tilde{Z}^1_1 + d\tilde{W}^0_0$. Lemma C.3 then implies that $\tilde{W}^0_0 = I'^0_0 + \delta m^0_1$ for some invariant I'^0_0. By the definition of $\tilde{W}^0_0$, this statement is equivalent to

$$\partial_{[\rho}Z_{0\,\mu_{[q]}|\,\nu_{[p-q]}]} = I'_{0\,[\mu_{[q]}\nu_{[p-q]}\rho]} + \delta m_{1\,[\mu_{[q]}\nu_{[p-q]}\rho]}\,.$$

Inserting this result into (C.36) yields

$$\alpha_{0\,\mu_{[q]}|\,\nu_{[p+1-q]}} - I'_{0\,[\mu_{[q]}\nu_{[p+1-q]}]} = \delta((-)^q Z_{1\,\mu_{[q]}|\,\nu_{[p+1-q]}} + m_{1\,[\mu_{[q]}\nu_{[p+1-q]}]}) - Z_{0\,\mu_{[q]}|\,[\nu_{[p-q]},\nu_{p+1-q}]}\,.$$

This equation has the same form as (C.33) and can be solved in the same way to get the following result:

$$\begin{aligned}
Z_{1\,\mu_{[q]}|\,\nu_{[p+1-q]}} &= (-)^{q+1}m_{1\,[\mu_{[q]}\nu_{[p+1-q]}]} + Z'_{1\,\mu_{[q]}|\,\nu_{[p+1-q]}} + \beta_{1\,\mu_{[q]}|\,[\nu_{[p-q]},\nu_{p+1-q}]} + \delta\beta_{2\,\mu_{[q]}|\,\nu_{[p+1-q]}}\\
&\quad + \frac{1}{s}\Big[T^q_{\rho_{[p+1]}}\frac{\partial^L R_{\mu_{[q]}}(K^{q+1},\tilde{\mathcal{H}})}{\partial K^{q+1}_{\rho_{[p+1]}}}\Big]_{1,\,\nu_{[p+1-q]}} + [P(\tilde{\mathcal{H}})]_{1,\,\nu_{[p+1-k]}}\,,\\
\alpha_{0\,\mu_{[q]}|\,\nu_{[p+1-q]}} &= I'_{0\,[\mu_{[q]}|\,\nu_{[p+1-q]}]} + (-)^q\delta(Z'_{1\,\mu_{[q]}|\,\nu_{[p+1-q]}}) + Z'_{0\,\mu_{[q]}|\,[\nu_{[p-q]},\nu_{p+1-q}]}\\
&\quad + [Q_{\mu_{[q]}}(K^{q+1})]_{\nu_{[p+1-q]}} + (-)^k[R(K^{q+1},\tilde{\mathcal{H}})]_{0,\,\nu_{[p+1-q]}}\,.
\end{aligned}$$

Removing the completely antisymmetric parts of these equations yields the desired result. $\square$

This ends the proof of Lemma C.1 for $k \geqslant q$.

C.3 Replacing all Z and Y by invariants

We will now prove the following lemma:

Lemma C.4. *The Euler-Lagrange derivatives of a_k can be written*

$$\begin{aligned}
\frac{\delta^L a_k}{\delta C^{*\,\mu_{[q]}}_{p+1}} &= (-)^{p+1}\delta(Z'_{k-p\;\mu_{[q]}})\,,\\
\frac{\delta^L a_k}{\delta C^{*\,\mu_{[q]}|\,\nu_{[p+1-j]}}_{j}} &= (-)^{j}\delta(Z'_{k+1-j\;\mu_{[q]}|\,\nu_{[p+1-j]}}) - Z'_{k-j\;\mu_{[q]}|\,[\nu_{[p-j]},\nu_{p+1-j}]}\quad, q<j\leqslant p\,,\\
\frac{\delta^L a_k}{\delta C^{*\,\mu_{[q]}|\,\nu_{[p+1-j]}}_{j}} &= (-)^{j}\delta(Z'_{k+1-j\;\mu_{[q]}|\,\nu_{[p+1-j]}}) - Z'_{k-j\;\mu_{[q]}|\,[\nu_{[p-j]},\nu_{p+1-j}]}\big|_{\,sym\,of\,C^*_j}\;, 1\leqslant j\leqslant q\,,\\
\frac{\delta^L a_k}{\delta \phi^{\mu_{[q]}|\,\nu_{[q]}}} &= \delta(Y'_{k+1\;\mu_{[q]}|\,\nu_{[q]}}) + \beta D_{\mu_{[q]}|\,\nu_{[q]}|\,\rho_{[p]}|\,\sigma_{[q]}} Z'^{\sigma_{[q]}|\,\rho_{[p]}}_{k}\,,
\end{aligned}$$

where Z'_l ($k-p\leqslant l\leqslant k$) and Y'_{k+1} are invariant polynomials, except in the following cases. When $k=p+1-m(q+1)$ for some strictly positive integer m, there is an additionnal term in the first non-trivial equation:

$$\frac{\delta^L a_k}{\delta C^{*\,\mu_{[q]}|\,\nu_{[p+1-k]}}_{k}} = (-)^k\delta Z'_{1\;\mu_{[q]}|\,\nu_{[p+1-k]}} - Z'_{0\;\mu_{[q]}|\,[\nu_{[p-k]},\nu_{p+1-k}]} + [Q_{\mu_{[q]}}(K^{q+1})]_{\nu_{[p+1-k]}}\big|_{\,sym\,of\,C^*_k}\,,$$

where Q is a polynomial of degree m in K^{q+1}. Furthermore, when $k=p+1-r(D-p-1)-s(q+1)$ for a couple of integer $r,s>0$, then there is an additional term in each Euler-Lagrange derivative:

$$\begin{aligned}
\frac{\delta^L a_k}{\delta C^{*\,\mu_{[q]}|\,\nu_{[p+1-j]}}_{j}} &= (-)^{j}\delta(Z'_{k+1-j\;\mu_{[q]}|\,\nu_{[p+1-j]}}) - Z'_{k-j\;\mu_{[q]}|\,[\nu_{[p-j]},\nu_{p+1-j}]}\big|_{\,sym\,of\,C^*_j}\\
&\quad +(-)^{k+p+1}A_{k-j}[R_{\mu_{[q]}}(K^{q+1},\tilde{\mathcal{H}})]_{k-j\;\nu_{[p+1-j]}}\big|_{\,sym\,of\,C^*_j}\\
\frac{\delta^L a_k}{\delta \phi^{\mu_{[q]}|\,\nu_{[q]}}} &= \delta(Y'_{k+1\;\mu_{[q]}|\,\nu_{[q]}}) + \beta D_{\mu_{[q]}|\,\nu_{[q]}|\,\rho_{[p]}|\,\sigma_{[q]}} Z'^{\sigma_{[q]}|\,\rho_{[p]}}_{k}\\
&\quad +A\,\delta^{[\sigma_{[q]}\alpha\mu_{[p]}\xi]}_{[\nu_{[q]}\beta\rho_{[p+1]}]}\partial_\alpha\partial^\beta(x_\xi\,[R_{\sigma_{[q]}}(K^{q+1},\tilde{\mathcal{H}})]^{\rho_{[p+1]}}_{k})\,,
\end{aligned}$$

where $A=\beta\frac{p+q+2}{(D-p-q-1)(p+1)!q!}A_k(-)^{p+k+1}$.

Proof: By Lemma C.1, we know that the Z's involved in the first non-trivial equation satisfy (C.32) and that this equation has the required form. We will proceed by induction

and prove that when Z_{k-j} (where $k-j \geqslant 1$) satisfies (C.32), then the equation for $\frac{\delta^L a_k}{\delta C_j^*}$ also has the desired form and Z_{k-j+1} also satisfies (C.32).

Let us assume that Z_{k-j} satisfies (C.32) and consider the following equation:

$$\frac{\delta^L a_k}{\delta C^*_{j\ \mu_{[q]}|\,\nu_{[p+1-j]}}} = (-)^j \delta(Z_{k+1-j}^{\mu_{[q]}|\,\nu_{[p+1-j]}}) - Z_{k-j}^{\mu_{[q]}|\,[\nu_{[p-j]},\nu_{p+1-j}]}|_{\,sym\,of\,C_j^*} \,. \tag{C.37}$$

Inserting (C.32) for Z_{k-j} into this equation yields

$$\begin{aligned}\frac{\delta^L a_k}{\delta C^*_{j\ \mu_{[q]}|\,\nu_{[p+1-j]}}} &= (-)^j \delta\Big(Z_{k+1-j}^{\mu_{[q]}|\,\nu_{[p+1-j]}} - \beta_{k-j+1}^{\mu_{[q]}|\,[\nu_{[p-j]},\nu_{p-j+1}]}|_{\,sym\,of\,C_j^*}\Big) \qquad \text{(C.38)}\\ &+ (-)^{k+p} a_{k-j} \delta\Big[P^{\mu_{[q]}}(\tilde{\mathcal{H}}) + \frac{1}{s} T^q_{\rho_{[p+1]}} \frac{\partial^L R^{\mu_{[q]}}(K^{q+1},\tilde{\mathcal{H}})}{\partial K^{q+1}_{\rho_{[p+1]}}}\Big]^{\nu_{[p+1-j]}}_{k-j+1}|_{\,sym\,of\,C_j^*}\\ &+ \Big(-Z'^{\,\mu_{[q]}|\,[\nu_{[p-j]},\nu_{p+1-j}]}_{k-j} + (-)^{p+k+1} A_{k-j}[R^{\mu_{[q]}}(K^{q+1},\tilde{\mathcal{H}})]^{\nu_{[p+1-j]}}_{k-j}\Big)|_{\,sym\,of\,C_j^*}\,.\end{aligned}$$

Note that one can omit to project on the symmetries of C^*_{j+1} when inserting (C.32) into (C.37). Indeed the Young components that are removed by this projection would be removed later anyway by the projection on the symmetries of C_j^*.

Defining the invariant

$$\begin{aligned} Z'^{\,\mu_{[q]}|\,\nu_{[p+1-j]}}_{k+1-j} \ &\equiv\ Z^{\mu_{[q]}|\,\nu_{[p+1-j]}}_{k+1-j}|_{\mathcal{N}=0}\\ &+(-)^{k+p+j} A_{k-j}\Big[P^{\mu_{[q]}}(\tilde{\mathcal{H}}) + \frac{1}{s} T^q_{\rho_{[p+1]}} \frac{\partial^L R^{\mu_{[q]}}(K^{q+1},\tilde{\mathcal{H}})}{\partial K^{q+1}_{\rho_{[p+1]}}}\Big]^{\nu_{[p+1-j]}}_{k-j+1}|_{\,sym\,of\,C_j^*}|_{\mathcal{N}=0}\end{aligned}$$

and setting $\mathcal{N}=0$ in the last equation yields, as β_{k-j+1} is at least linear in $\mathcal{N}$,

$$\begin{aligned}\frac{\delta^L a_k}{\delta C^*_{j\ \mu_{[q]}|\,\nu_{[p+1-j]}}} &= (-)^j \delta(Z'^{\,\mu_{[q]}|\,\nu_{[p+1-j]}}_{k+1-j}) - Z'^{\,\mu_{[q]}|\,[\nu_{[p-j]},\nu_{p+1-j}]}_{k-j}|_{\,sym\,of\,C_j^*}\\ &+(-)^{p+k+1} A_{k-j}[R^{\mu_{[q]}}(K^{q+1},\tilde{\mathcal{H}})]^{\nu_{[p+1-j]}}_{k-j}|_{\,sym\,of\,C_j^*}\,. \end{aligned}\tag{C.39}$$

This proves the part of the induction regarding the equations for the Euler-Lagrange derivatives. We now prove that Z_{k-j+1} verifies (C.32).

Substracting (C.39) from (C.38), we get

$$\begin{aligned} 0 \ &=\ (-)^j \delta\Big(Z^{\mu_{[q]}|\,\nu_{[p+1-j]}}_{k+1-j} - Z'^{\,\mu_{[q]}|\,\nu_{[p+1-j]}}_{k+1-j} - \beta^{\mu_{[q]}|\,[\nu_{[p-j]},\nu_{p+1-j}]}_{k+1-j}|_{\,sym\,of\,C_j^*}\\ &+(-)^{j+k+p} A_{k-j}\Big[P^{\mu_{[q]}}(\tilde{\mathcal{H}}) + \frac{1}{s} T^q_{\rho_{[p+1]}} \frac{\partial^L R^{\mu_{[q]}}(K^{q+1},\tilde{\mathcal{H}})}{\partial K^{q+1}_{\rho_{[p+1]}}}\Big]^{\nu_{[p+1-j]}}_{k+1-j}|_{\,sym\,of\,C_j^*}\Big)\,.\end{aligned}$$

As $k+1-j>0$, this implies

$$\begin{aligned} Z^{\mu_{[q]}|\,\nu_{[p+1-j]}}_{k+1-j} \ &=\ Z'^{\,\mu_{[q]}|\,\nu_{[p+1-j]}}_{k+1-j} + (-)^{j-1}\delta\beta^{\mu_{[q]}|\,\nu_{[p+1-j]}}_{k-j} + \beta^{\mu_{[q]}|\,[\nu_{[p-j]},\nu_{p+1-j}]}_{k-j+1}|_{\,sym\,of\,C_j^*}\\ &+A_{k+1-j}\Big[P^{\mu_{[q]}}(\tilde{\mathcal{H}}) + \frac{1}{s} T^q_{\rho_{[p+1]}} \frac{\partial^L R^{\mu_{[q]}}(K^{q+1},\tilde{\mathcal{H}})}{\partial K^{q+1}_{\rho_{[p+1]}}}\Big]^{\nu_{[p+1-j]}}_{k+1-j}|_{\,sym\,of\,C_j^*}\,,\end{aligned}$$

which is the expression (C.32) for Z_{k+1-j}.

Assuming that Z_{k-j} satisfies (C.32) , we have thus proved that the equation for $\frac{\delta^L a_k}{\delta C^*_j}$ has the desired form and that Z_{k+1-j} also satisfies (C.32). Iterating this step, one shows that all Z's satisfy (C.32) and that the equations involving only Z's have the desired form.

It remains to be proved that the Euler-Lagrange derivative with respect to the field takes the right form. Inserting the expression (C.32) for Z_k into (C.30) and some algebra yield

$$\begin{aligned}\frac{\delta^L a_k}{\delta \phi^{\mu_{[q]}|\,\nu_{[q]}}} &= \delta(\tilde{Y}_{k+1\;\mu_{[q]}|\,\nu_{[q]}}|_{\,sym\,of\,\phi}) + \beta D_{\mu_{[q]}|\,\nu_{[q]}|\,\rho_{[p]}|\,\sigma_{[q]}} {Z'}_k^{\sigma_{[q]}|\,\rho_{[p]}} \\ &\quad + A\,\delta^{[\sigma_{[q]}\alpha\mu_{[p]}\xi]}_{[\nu_{[q]}\beta\rho_{[p+1]}]}\partial_\alpha\partial^\beta(x_\xi\,[R_{\sigma_{[q]}}(K^{q+1},\tilde{\mathcal{H}})]_k^{\rho_{[p+1]}})|_{\,sym\,of\,\phi}\,,\end{aligned}$$

where

$$\begin{aligned}\tilde{Y}_{k+1\;\mu_{[q]}|\,\nu_{[q]}} &\equiv Y_{k+1\;\mu_{[q]}|\,\nu_{[q]}} + \beta D_{\mu_{[q]}|\,\nu_{[q]}|\,\rho_{[p]}|\,\sigma_{[q]}}\beta_{k+1}^{\sigma_{[q]}|\,\rho_{[p]}} \\ &\quad + c\,\delta^{[\sigma_{[q]}\alpha\mu_{[p]}]}_{[\nu_{[q]}\beta\rho_{[p]}]}\partial_\alpha\Big[P_{\sigma_{[q]}}(\tilde{\mathcal{H}}) + \frac{1}{s}T^q_{\lambda_{[p+1]}}\frac{\partial^L R^{\sigma_{[q]}}(K^{q+1},\tilde{\mathcal{H}})}{\partial K^{q+1}_{\lambda_{[p+1]}}}\Big]_{k+1}^{[\rho_{[p]}\beta]} \\ &\quad + (-)^{k+q+1}A\,\delta^{[\sigma_{[q]}\alpha\mu_{[p]}\xi]}_{[\nu_{[q]}\beta\rho_{[p+1]}]}\partial_\alpha(x_\xi\,[R_{\sigma_{[q]}}(K^{q+1},\tilde{\mathcal{H}})]_{k+1}^{[\rho_{[p+1]}\beta]})\end{aligned}$$

and $c \equiv \beta\frac{1}{(p+1)!q!}A_k(-)^{p+k+1}$. Defining $Y'_{k+1\;\mu_{[p]}|\,\nu_{[q]}} \equiv \tilde{Y}_{k+1\;\mu_{[q]}|\,\nu_{[q]}}|_{\,sym\,of\,\phi}|_{\mathcal{N}=0}$ and setting $\mathcal{N}=0$ in the above equation completes the proof of Lemma C.4. $\Box$

C.4 Euler-Lagrange derivative with respect to the field

In this section, we manipulate the Euler-Lagrange derivative of a_k with respect to the field ϕ .

We have proved in the previous section that it can be written in the form

$$\begin{aligned}\frac{\delta^L a_k}{\delta \phi^{\mu_{[p]}|\,\nu_{[q]}}} &= \delta(Y'_{k+1\;\mu_{[p]}|\,\nu_{[q]}}) + \beta D_{\mu_{[p]}|\,\nu_{[q]}|\,\rho_{[p]}|\,\sigma_{[q]}} {Z'}_k^{\sigma_{[q]}|\,\rho_{[p]}} \\ &\quad + A\,\delta^{[\sigma_{[q]}\alpha\mu_{[p]}\xi]}_{[\nu_{[q]}\beta\rho_{[p+1]}]}\partial_\alpha\partial^\beta(x_\xi\,[R_{\sigma_{[q]}}(K^{q+1},\tilde{\mathcal{H}})]_k^{\rho_{[p+1]}})|_{\,sym\,of\,\phi}\,.\end{aligned}$$

As a_k is invariant, it can depend on $\phi_{\mu_{[p]}|\,\nu_{[q]}}$ only through $K_{\mu_{[p]}\alpha|\,\nu_{[q]}\beta}$, which implies that $\frac{\delta^L a_k}{\delta\phi^{\mu_{[p]}|\,\nu_{[q]}}} = \partial^{\alpha\beta}X_{[\mu_{[p]}\alpha]|\,[\nu_{[q]}\beta]}$, where X has the symmetry of the curvature. This in turn implies that $\delta(Y'_{k+1\;\mu_{[p]}|\,\nu_{[q]}}) = \partial^{\alpha\beta}W_{\mu_{[p]}\alpha|\,\nu_{[q]}\beta}$ for some W with the Young symmetry $[p+1,q+1]$. Let us consider the indices $\mu_{[p]}$ as form indices. As $H^{D-p}_{k+1}(\delta|\,d) \cong H^D_{p+1+k}(\delta|\,d) \cong 0$ for $k>0$, the last equation implies

$$Y'_{k+1\;\mu_{[p]}|\,\nu_{[q]}} = \delta A_{k+2\;\mu_{[p]}|\,\nu_{[q]}} + \partial^\lambda T_{k+1\;[\lambda\mu_{[p]}]|\,\nu_{[q]}}\,. \tag{C.40}$$

By the induction hypothesis for $p+1+k$, we can take A_{k+2} and T_{k+1} invariant. Antisymmetrizing (C.40) over the indices $\mu_q\ldots\mu_p\nu_1\ldots\nu_q$ yields

$$0 = \delta A_{k+2\;\mu_1\ldots\mu_{q-1}[\mu_q\ldots\mu_p|\,\nu_1\ldots\nu_q]} + \partial^\lambda T_{k+1\;\lambda\mu_1\ldots\mu_{q-1}[\mu_q\ldots\mu_p|\,\nu_1\ldots\nu_q]}\,.$$

The solution of this equation for T_{k+1} is

$$T_{k+1\,\mu_0\dots\mu_{q-1}[\mu_q\dots\mu_p|\,\nu_1\dots\nu_q]} = \delta Q_{k+2\,\mu_0\dots\mu_{q-1}|\,[\mu_q\dots\mu_p\nu_1\dots\nu_q]} + \partial^\alpha S_{k+1\,\alpha\mu_0\dots\mu_{q-1}|\,[\mu_q\dots\mu_p\nu_1\dots\nu_q]} + \Big[U^{(u)}_{[\mu_q\dots\mu_p\nu_1\dots\nu_q]}(\tilde{\mathcal{H}})\Big]^{\rho_{[D-q]}}_{k+1}\epsilon_{\mu_0\dots\mu_{q-1}\rho_{[D-q]}}\,,$$

where $U^{(u)}$ is a polynomial of degree u in $\tilde{\mathcal{H}}$, present when $k+q+1=D-u(D-p-1)$ for some strictly positive integer u. As T and $U^{(u)}(\tilde{\mathcal{H}})$ are invariant, we can use the induction hypothesis for $k'=k+1+q$. This implies

$$\begin{aligned} T_{k+1\,\mu_0\dots\mu_{q-1}[\mu_q\dots\mu_p|\,\nu_1\dots\nu_q]} &= \delta Q'_{k+2\,\mu_0\dots\mu_{q-1}|\,[\mu_q\dots\mu_p\nu_1\dots\nu_q]} + \partial^\alpha S'_{k+1\,\alpha\mu_0\dots\mu_{q-1}|\,[\mu_q\dots\mu_p\nu_1\dots\nu_q]} \\ &+ \Big[U^{(u)}_{[\mu_q\dots\mu_p\nu_1\dots\nu_q]}(\tilde{\mathcal{H}}) + V^{(v,w)}_{[\mu_q\dots\mu_p\nu_1\dots\nu_q]}(K^{q+1},\tilde{\mathcal{H}})\Big]^{\rho_{[D-q]}}_{k+1}\epsilon_{\mu_0\dots\mu_{q-1}\rho_{[D-q]}}\,, \end{aligned} \tag{C.41}$$

where Q'_{k+2} and S'_{k+1} are invariants and $V^{(v,w)}$ is a polynomial of order v and w in K^{q+1} and $\tilde{\mathcal{H}}$ respectively, present when $D-q=v(q+1)+w(D-p-1)+k+1$ for some strictly positive integers v,w.

We define the invariant tensor $E_{\alpha\mu_{[p]}|\,\beta\nu_{[q]}}$ with Young symmetry $[p+1,q+1]$ by

$$E_{\alpha\mu_{[p]}|\,\beta\nu_{[q]}} = \sum_{i=0}^{q+1}\alpha_i S'_{k+1\;\rho_0\dots\rho_{i-1}[\nu_i\dots\nu_q|\,\beta\nu_1\dots\nu_{i-1}]\rho_i\dots\rho_p}\,\delta^{[\rho_0\dots\rho_p]}_{[\alpha\mu_{[p]}]}$$

where $\alpha_i=\alpha_0\frac{(q+1)!}{(q+1-i)!\,i!}$ and $\alpha_0=(-)^{pq}\frac{((p+1)!)^2}{(p-q)!\,(q!)^2\,(p-q+1)\,(p+2)\,\sum_{j=0}^{q}\frac{(p-j)!}{(q-j)!}}$.

Writing $\partial^{\alpha\beta}E_{k+1\;\alpha\mu_{[p]}|\,\beta\nu_{[q]}}$ in terms of S'_{k+1} and using (C.41) and (C.40) yields

$$\begin{aligned} Y'_{k+1\;\mu_{[p]}|\,\nu_{[q]}} &= \partial^{\alpha\beta}E_{k+1\;\alpha\mu_{[p]}|\,\beta\nu_{[q]}} + \delta F_{k+2\;\mu_{[p]}|\,\nu_{[q]}} \\ &\quad +\partial^\alpha\sum_{i=0}^{q}\beta_i\Big[V^{(v,w)}_{[\alpha\nu_{[i]}\mu_{i+1}\dots\mu_p]}(K^{q+1},\tilde{\mathcal{H}})\Big]^{\rho_{[D-q]}}_{k+1}\epsilon_{\mu_{[i]}\nu_{i+1}\dots\nu_q\rho_{[D-q]}}\,, \end{aligned} \tag{C.42}$$

where F_{k+2} is invariant, $\beta_i\equiv\alpha_0\frac{(p+2)q!}{(p+1)\,i!\,(q-i)!}$ and v is allowed to take the value $v=0$ to cover also the case of the polynomial $U^{(w)}(\tilde{\mathcal{H}})$.

C.5 Homotopy formula

We will now use the homotopy formula to reconstruct a_k from its Euler-Lagrange derivatives:

$$a_k^D = \int_0^1 dt\Big[\phi_{\mu_{[p]}|\,\nu_{[q]}}\frac{\delta^L a_k}{\delta\phi_{\mu_{[p]}|\,\nu_{[q]}}} + \sum_{j=1}^{p+1}C^*_{j\;\mu_{[q]}|\,\nu_{[p+1-j]}}\frac{\delta^L a_k}{\delta C^*_{j\;\mu_{[q]}|\,\nu_{[p+1-j]}}}\Big]\,d^Dx\,.$$

Inserting the expressions for the Euler-Lagrange derivatives given by Lemma C.4 yields

$$a_k^D = \int_0^1 dt\Big[\delta(\phi_{\mu_{[p]}|\,\nu_{[q]}}Y'^{\,\mu_{[p]}|\,\nu_{[q]}}_{k+1}) + \sum_{j=1}^{p+1}\delta(C^*_{j\;\mu_{[q]}|\,\nu_{[p+1-j]}}Z'^{\,\mu_{[q]}|\,\nu_{[p+1-j]}}_{k+1-j})$$

$$
\begin{aligned}
&+\sum_{j=1}^{k} C^{*}_{j\;\mu_{[q]}|\,\nu_{[p+1-j]}}(-)^{k+p+1}A_{k-j}[R^{\mu_{[q]}}(K^{q+1},\tilde{\mathcal{H}})]^{\nu_{[p+1-j]}}_{k-j}\\
&+\phi_{\mu_{[p]}|\,\nu_{[q]}}A\,\delta^{[\sigma_{[q]}\alpha\mu_{[p]}\xi]}_{[\nu_{[q]}\beta\rho_{[p+1]}]}\partial_\alpha\partial^\beta(x_\xi\,[R_{\sigma_{[q]}}(K^{q+1},\tilde{\mathcal{H}})]^{\rho_{[p+1]}}_{k})\\
&+C^{*}_{k\;\mu_{[q]}|\,\nu_{[p+1-k]}}[Q^{(m)\,\mu_{[q]}}(K^{q+1})]^{\nu_{[p+1-k]}}\Big]d^Dx+dn^{D-1}_{k}\,.
\end{aligned}
$$

Using the result (C.42) for Y'_{k+1} and some algebra, one finds

$$
\begin{aligned}
a^D_k \;=\; &\int_0^1 dt\Big[\delta(K_{\mu_{[p+1]}|\nu_{[q+1]}}E^{\mu_{[p+1]}|\nu_{[q+1]}}_{k+1}d^Dx)+a_v\,K^{q+1}_{\mu_{[p+1]}}[V^{(v,w)\,\mu_{[p+1]}}(K^{q+1},\tilde{\mathcal{H}})]^{D-q-1}_{k}\\
&+\sum_{j=1}^{p+1}\delta(C^{*}_{j\;\mu_{[q]}|\,\nu_{[p+1-j]}}Z'^{\,\mu_{[q]}|\,\nu_{[p+1-j]}}_{k+1-j}d^Dx)+a_r[\tilde{\mathcal{H}}^{\sigma_{[q]}}\,R_{\sigma_{[q]}}(K^{q+1},\tilde{\mathcal{H}})]^D_k\\
&+a_q\,[\tilde{\mathcal{H}}^{\sigma_{[q]}}]^{D-m(q+1)}_{k}\,Q^{(m)}_{\sigma_{[q]}}(K^{q+1})\Big]+d\bar{n}^{D-1}_{k}\,,
\end{aligned}
$$

where $a_v=(-)^{k(q+1)}\sum_{i=0}^{q}\beta_i\frac{i!(p-i)!}{p!}$, $a_r=(-)^{D(p+k+1)+\frac{p(p+1)+k(k+1)}{2}}$ and $a_q=(-)^k a_r$. In short,

$$
a^D_k\;=\;[P(K^{q+1},\tilde{\mathcal{H}})]^D_k+\delta\mu^D_{k+1}+d\bar{n}^{D-1}_k
$$

for some invariant μ^D_{k+1}, and some polynomial P of strictly positive order in K^{q+1} and $\tilde{\mathcal{H}}$.

We still have to prove that $\bar{n}^{D-1}_k$ can be taken invariant.
Acting with γ on the last equation yields $d(\gamma\bar{n}^{D-1}_k)=0$. By the Poincaré lemma, $\gamma\bar{n}^{D-1}_k=d(r^{D-2}_k)$. Furthermore, a well-known result on $H(\gamma|\,d)$ for positive antighost number k (see e.g. Appendix A.1 of [17]) states that one can redefine $\bar{n}^{D-1}_k$ in such a way that $\gamma\bar{n}^{D-1}_k=0$. As the pureghost number of $\bar{n}^{D-1}_k$ vanishes, the last equation implies that $\bar{n}^{D-1}_k$ is an invariant polynomial.

This completes the proof of Theorem 7.2 for $k\geqslant q$. □

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